From Euler-Lagrange equations to twist maps
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Alain Chenciner
Université Paris 7 & IMCCE
alain.chenciner@obspm.fr
https://perso.imcce.fr/alain-chenciner/

Abstract

Starting with elementary calculus of variations and Legendre transform, we shall see how the mathematical structures of conservative dynamics (Poincaré-Cartan integral invariant, symplectic structure, Hamiltonian form of the equations) arise from the simple computation of the variations of an action integral. Integrable and close to integrable geodesic flows on the 2-torus on the one hand, the planar circular restricted three-body problem in the lunar case on the second hand, will be taken as examples of hamiltonian systems with two degrees of freedom for which, thanks to the existence of a global "surface of section", the dynamics can be reduced to the iteration of a "twist map" of an annulus. Other sources as normal forms and periodic Hamiltonians are also discussed. Such maps are the best setting for introducing fundamental features of Hamiltonian systems : the periodic orbits stemming from the Poincaré-Birkhoff fixed point theorem, the Aubry-Mather sets, simplest examples of weak K.A.M. solutions, and the quasi-periodic motions consequence of the Moser invariant curve theorem, simplest examples of K.A.M. solutions.

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1 The Euler-Lagrange equations

1.1 Introduction

Classical mechanics (see [A]) deals in general with second order ordinary differential equations of the form

$$\ddot{q} = F(q, \dot{q}). \tag{E_1}$$

The terms depending on the velocities $\dot{q}$ are termed "dissipative": they correspond to frictions (damping) or excitations. In their absence, one gets "conservative" equations $\ddot{q} = F(q)$ which are often of the form

$$\ddot{q} = \nabla U(q), \tag{E_2}$$

where U is a "potential function" and the gradient is relative to some Riemannian metric g on the configuration space, which defines a "kinetic energy". The paradigmatic example is "the n-body problem", where the configuration $q = (\vec{r}_1, \vec{r}_2, \dots, \vec{r}_n) \in E^n$ is the set of positions of n point masses in an euclidean space E and the equations are

$$\ddot{\vec{r}}_i = g \sum_{j \neq i} \frac{m_j(\vec{r}_j - \vec{r}_i)}{||\vec{r}_i - \vec{r}_j||_E^3}.$$

Here the m_i are positive masses, the potential function is

$$U(q) = \sum_{i<j} \frac{m_i m_j}{||\vec{r}_i - \vec{r}_j||_E}$$

and the Riemannian metric g is defined by the (constant) scalar product

$$g(r,s) = < r, s >_g = \sum_{i=1}^{n} m_i \langle \vec{r}_i, \vec{s}_i \rangle_E \quad \text{if} \quad r = (\vec{r}_1, \vec{r}_2, \dots, \vec{r}_n),\ s = (\vec{s}_1, \vec{s}_2, \dots, \vec{s}_n).$$

Such equations are known, since Lagrange, to be the so-called Euler-Lagrange equations of an "action functional", the Lagrangian action $\int L\big(q(t), \dot{q}(t)\big)dt$, where the *Lagrangian*

$$L(q, \dot{q})) = \frac{1}{2}||\dot{q}||_g^2 + U(q),$$

is the difference between the kinetic energy $\frac{1}{2}||\dot{q}||_g^2$ and the potential energy $-U(q)$. This means that the solutions of (E_2) are exactly the set of "extremal" curves of the action functional. It is the mathematical formulation of the so-called *principle of least action*. In the case when $U \equiv 0$, one gets the "geodesics" of the Riemannian metric g. This origin makes natural the following "convexity" hypotheses:

Warning. *The configuration space will be noted M. It could be an arbitrary n-dimensional differentiable manifold but, for convenience, the reader may suppose that it is either an open subset of $\mathbb{R}^n$ or the n-torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$. This is in particular assumed when we work with global coordinates on the tangent bundle TM which, in such cases is canonically identified with the product $M \times \mathbb{R}^n$.*

General convexity hypotheses. The C^∞ (C^3 would be enough), possibly time-dependent, Lagrangian $L(q, \dot{q}, t)$

$$L : TM \times \mathbb{R} = M \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$$

will be assumed to satisfy the "Tonelli" hypotheses which insure the existence of minimizers under natural hypotheses of coercivity[1]:

1) L is strictly convex in $\dot{q}$, that is (in the sense of quadratic forms) :

$$\forall q, \dot{q}, t,\ \frac{\partial^2 L}{\partial \dot{q}^2}(q, \dot{q}, t) > 0;$$

Note that this makes sense intrinsically because the derivatives of the restriction of L to the vector space T_qM are well defined.

2) L is superlinear in $\dot{q}$:

$$\forall C \in \mathbb{R}, \exists D \in \mathbb{R},\ \forall q, \dot{q}, t,\ L(q, \dot{q}, t) \geq C||\dot{q}|| - D,$$

that is $\lim_{||\dot{q}|| \to \infty} \frac{L(q,\dot{q},t)}{||\dot{q}||} = +\infty$ *uniformly in* (q, t).

1.2 The fundamental computation

The whole structure of classical conservative mechanics is the consequence of a single computation, the one giving the variation of the action $\mathcal{A}_L(\gamma)$ of a path $\gamma : [a, b] \to M,\ t \mapsto q(t)$ (the reader may suppose that $M = \Omega$ is an open subset of $\mathbb{R}^n$):

$$\mathcal{A}_L(\gamma) = \int_a^b L(q(t), \dot{q}(t), t)\, dt, \quad \text{where} \quad \dot{q}(t) = \frac{dq}{dt}(t),$$

under an *arbitrary* variation of the path where neither the end-points nor the interval of variation of the parameter are fixed. Let us start with a regular (say at least C^2) path γ and consider a variation of γ, that is a family of paths

$$\gamma_u : [a(u), b(u)] \to M, \quad t \mapsto q(u, t), \quad u \in]-\epsilon, +\epsilon[, \quad \gamma_0 = \gamma,$$

regular with respect to both variables (u, t) (these regularity hypotheses will soon be weakened). The *infinitesimal variation* is by definition the vector field on M along γ defined by

$$X(t) = X_0(t) = \frac{\partial q}{\partial u}(0, t).$$

[1] there are of course important situations where convexity does not hold, for example Lorentzian metrics

It plays the role of a tangent vector at γ to the "manifold" of paths. More generally, we shall use the following notations:

$$X_u(t) = \frac{\partial q}{\partial u}(u,t), \quad \dot{q}(u,t) = \frac{\partial q}{\partial t}(u,t).$$

While $\dot{q}(u,t)$ is simply the velocity at time t of the path γ_u, the vector field $t \mapsto X_u(t)$ along γ_u, is the velocity at u of the "path of paths" $u \mapsto \gamma_u$.

Computing the derivative of the function $u \mapsto \mathcal{A}_L(\gamma_u)$ via an integration by parts, one gets the following

Fundamental formula:

$$\begin{aligned}
&\frac{d}{du}\Big(\mathcal{A}_L(\gamma_u)\Big) = \frac{d}{du}\int_{a(u)}^{b(u)} L\Big(q(u,t),\dot{q}(u,t),t\Big)\,dt \\
&= \int_{a(u)}^{b(u)} \left[\frac{\partial L}{\partial q}\big(q(u,t),\dot{q}(u,t),t\big) - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\big(q(u,t),\dot{q}(u,t),t\big)\right)\right]\cdot X_u(t)\,dt \\
&+ \frac{\partial L}{\partial \dot{q}}\big(q(u,t),\dot{q}(u,t),t\big)\cdot X_u(t)\,\Big|_{t=b(u)} - \frac{\partial L}{\partial \dot{q}}\big(q(u,t),\dot{q}(u,t),t\big)\cdot X_u(t)\,\Big|_{t=a(u)} \\
&+ L\big(q(u,t),\dot{q}(u,t),t\big)\frac{db}{du}(u)\,\Big|_{t=b(u)} - L\big(q(u,t),\dot{q}(u,t),t\big)\frac{da}{du}(u)\,\Big|_{t=a(u)},
\end{aligned}$$

a formula that we shall abreviate in

$$\frac{d\mathcal{A}_L}{du} = \int_a^b \left(\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}}\right)\cdot\frac{\partial q}{\partial u}\,dt \;+\; \left[\frac{\partial L}{\partial \dot{q}}\cdot\frac{\partial q}{\partial u} + L\frac{dt}{du}\right]_a^b.$$

Restricting to variations parametrized by a fixed interval $[a,b]$ and such that the end-points are fixed, i.e. $q(u,a) = q(0,a)$ and $q(u,b) = q(0,b)$, one gets the classical *Euler-Lagrange* equations for the *extremals*, that is the paths γ such that $d\mathcal{A}_L(\gamma)X = 0$ for any infinitesimal variation X with fixed interval and fixed end points (i.e. $X(a) = 0$ and $X(b) = 0$):

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\big(q(t),\dot{q}(t),t\big)\right) = \frac{\partial L}{\partial q_i}\big(q(t),\dot{q}(t),t\big), \quad i = 1,\cdots,n. \qquad (E)$$

Such writing supposes of course that we have global coordinates $q = (q_1,\cdots,q_n)$ in $M = \Omega \subset \mathbb{R}^n$. We shall give soon an intrinsic interpretation of these equations. In order to put equations E into a nice "explicit" form, we notice that the "general hypotheses" we made on L imply that the *Legendre mapping*

$$\Lambda : TM \times \mathbb{R} \to T^*M \times \mathbb{R}$$

defined intrinsically (same reason as for the convexity hypotheses) by

$$\Lambda(q,\dot{q},t) = (q,p,t), \quad p = \frac{\partial L}{\partial \dot{q}}(q,\dot{q},t),$$

is a global diffeomorphism (strict convexity for all p of $\dot{q} \mapsto L(q,\dot{q},t) - p\cdot\dot{q}$ implies the injectivity of Λ and surlinearity implies that it is proper, hence surjective). One says that L is *globally regular*. From this, two important results follow:

1) Regularity of extremals: any extremal is as regular as L. This means that if we had assumed paths to be only C^0 and piecewise C^1, and the variations accordingly, the extremals among paths with fixed interval of definition and fixed ends would still be as regular as the Lagrangian. The proof consists in exchanging the roles of the two terms in the integration by parts: this leads to the integral form of the equations

$$\frac{\partial L}{\partial \dot{q}_i}\big(q(t), \dot{q}(t), t\big) = \int_a^t \frac{\partial L}{\partial q_i}\big(q(s), \dot{q}(s), s\big)ds + C_i\,,$$

where the C_i are constants and implies the regularity by a *bootstrap* argument: precisely, $q(t)$ being C^0 and piecewise C^1, the above equations imply that $p(t) = \frac{\partial L}{\partial \dot{q}_i}\big(q(t), \dot{q}(t), t\big)$ is continuous. Hence $\Lambda(q(t), \dot{q}(t), t) = (q(t), p(t), t)$ is continuous, which because Λ is a homeorphism implies the continuity of $\dot{q}(t)$, and so on by induction.
This justifies our working only with regular paths. In the case of minimizers, one could even work with absolutely continuous paths. In fact, small enough extremals are minimizers and their regularity amounts, as in the case of straight lines, to the remark that a broken curve can always be shortened by smoothing the angle.

2) Existence of the Euler-Lagrange flows: it follows from the fact that Λ is a diffeomorphism that equations (E) define a (time-dependant if L is) vector field X_L in TM : in coordinates these equations take the "mixed" form

$$\frac{dp_i}{dt} = \frac{\partial L}{dq_i}(q, \dot{q}, t),\ \frac{dq_i}{dt} = \dot{q}_i.$$

One concludes by using Λ^{-1} as a change from the (q, p, t) to the $(q, \dot{q}, t)$. More computationally, in coordinates, one can use the invertibility at each point of the Hessian matrix $\left(\frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j}\right)$ to transform the implicit differential equations (E) into explicit ones of the form

$$\ddot{q}_i = f(q, \dot{q}, t).$$

We end this section with two important consequences of the fundamental computation and some remarks on "parametric Lagrangians": independent **3) Internal variations and the energy:** Suppose L and c are of class C^2, and let us consider only variations of the form

$$q(u, t) = q\big(t + u\xi(t)\big), \quad \xi(a) = \xi(b) = 0,$$

that is parameter changes (sometimes called *internal variations*).
As $X(t) = \frac{\partial q}{\partial u}(0, t) = \dot{q}(t)\xi(t)$ and $\xi(t)$ is arbitrary, one deduces from Euler-Lagrange equations that the extremality condition with respect to such variations reads

$$\left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\big(q(t), \dot{q}(t), t\big)\right) - \frac{\partial L}{\partial q}\big(q(t), \dot{q}(t), t\big)\right] \cdot \dot{q}(t) = 0,$$

that is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot q}\big(q(t),\dot q(t),t\big)\cdot \dot q(t) - L\big(q(t),\dot q(t),t\big)\right) = -\frac{\partial L}{\partial t}\big(q(t),\dot q(t),t\big).$$

This is the *conservation of the energy* $H = p\cdot \dot q - L$ by the extremals of an *autonomous* (i.e. independent of time) Lagrangien $L(q,\dot q)$. We shall come back to this fundamental point.

4) Symmetries and first integrals (Noether's theorem): Let X be a vector field on M, $L : TM\times\mathbb{R}$ a Lagrangian. Suppose that L is left invariant under the (possibly local) flow φ_u of X, that is for all q,t,u such that the left-hand side is defined,

$$L\big(\varphi_u(q), d\varphi_u(q)\dot q, t\big) = L(q,\dot q,t).$$

Then the following expression is a *first integral* of the Euler-Lagrange equations, which means that *it remains constant along any extremal* $q(t)$:

$$(p\cdot X)(t) = \frac{\partial L}{\partial \dot q}\,(q(t),\dot q(t),t)\cdot X\big(q(t)\big).$$

For the proof, one considers the family of curves $q(u,t) = \varphi_u\big(q(t)\big)$. They are extremals of L and the action $\mathcal{A}(u)$ is independent of u, which gives the result. In the next section we shall encounter the trivial case of this theorem where L does not depend on one of the variables q_i (one says that q_i is *ignorable*) and $p.X$ is simply $\frac{\partial L}{\partial \dot q_i}$.

5) Parametric Lagrangians and again the energy: These are the Lagrangians which are homogeneous of degree 1 in the velocities $\dot q_i$, as is for example the integrand $||\dot q||$ of the *length integral*. The formula ruling the change of variable in an integral implies that the parametrization (the *gauge*) of the extremals is not fixed. Such Lagrangians appear naturally, on the one hand when integrating 1-forms (see later), on the other hand when one seeks to transform a time-dependant Lagrangian $L(q,\dot q,t)$ defined on M, into a time independent Lagrangian $\mathcal{L}(q,\dot q,q_{n+1},\dot q_{n+1})$ defined on $M\times\mathbb{R}_+$ by setting

$$\mathcal{L}(q,q_{n+1},\dot q,\dot q_{n+1}) = L(q,\frac{1}{\dot q_{n+1}}\dot q, q_{n+1})\dot q_{n+1}.$$

Let $\Lambda : TM\times\mathbb{R}\to T^*M\times\mathbb{R}$ and $\Xi : T(M\times\mathbb{R})\to T^*(M\times\mathbb{R})$ be the Legendre mappings respectively associated to L and $\mathcal{L}$:

$$\Lambda(q,\dot q,t) = (q,p,t),\quad \Xi(q,q_{n+1},\dot q,\dot q_{n+1}) = (q,q_{n+1},P,P_{n+1}).$$

One has

$$\Lambda(q,\frac{1}{\dot q_{n+1}}\dot q, q_{n+1}) = (q,P,q_{n+1}).$$

The homogeneity of $\mathcal{L}$ in the velocities implies the Euler identity

$$\mathcal{L}(q,q_{n+1},\dot q,\dot q_{n+1}) = P\cdot\dot q + P_{n+1}\dot q_{n+1},.$$

After dividing by $\dot{q}_{n+1}$, this becomes

$$P_{n+1} + [p \cdot \dot{q} - L](q, \frac{1}{\dot{q}_{n+1}}\dot{q}, q_{n+1}) = 0.$$

Supposing that L is regular and introducing the total energy (the *Hamiltonian*)

$$H(q, p, t) = [p \cdot \dot{q} - L] \circ \Lambda^{-1}(p, q, t),$$

we get for $\mathcal{L}$ the following *constraint* making clear the fact that Ξ is not of maximal rank (non regularity of $\mathcal{L}$) :

$$P_{n+1} + H(q, P, q_{n+1}) = 0.$$

The extremals of L can be recovered from the solutions of the Euler-Lagrange equations associated to $\mathcal{L}$, that is

$$\frac{dP_{n+1}}{d\tau} = \frac{\partial L}{\partial t}(q, \frac{1}{\dot{q}_{n+1}}\dot{q}, q_{n+1})\dot{q}_{n+1}, \quad \frac{dP}{d\tau} = \frac{\partial L}{\partial q}(q, \frac{1}{\dot{q}_{n+1}}\dot{q}, q_{n+1})\dot{q}_{n+1}.$$

(Notice that the first equation is equivalent to the one obtained above by looking at the internal variations.) Indeed, if $\big(q(\tau), q_{n+1}(\tau)\big)$ is an extremal of $\mathcal{L}$, the curve $t \mapsto q\big(\tau(t)\big)$ defined by setting $t = q_{n+1}(\tau)$ (i.e. fixing the gauge), is an extremal of L. Conversely, to any extremal $q(t)$ of L corresponds the extremal $\big(q(\tau), \tau\big)$ of $\mathcal{L}$.

Transforming L into $\mathcal{L}$ allows applying Noether's theorem to space-time symmetries of a Lagrangian, i.e. to a vector field X on $M \times \mathbb{R}_+$. We can recover in this way the conservation of the energy for the autonomous Lagrangians as a consequence of the invariance of the Lagrangian under time translations.

6) Parametric Lagrangians again and the Cartan formula: Let V be a manifold, ϖ a differential 1-form on V. To each compact oriented regular curve (with boundary) (i.e. oriented 1-dimensional submanifold) Γ of V, we attach the real number $\mathcal{A}(\Gamma) = \int_\Gamma \varpi$. If $u \mapsto \Gamma_u$ is a differentiable family of such curves, we shall denote by $\mathcal{A}(u)$ the function $u \mapsto \mathcal{A}\big(\Gamma_u\big)$.

Definition 1 *The curve* Γ *is* stationary *for* $\mathcal{A}$ *if* $\frac{d\mathcal{A}}{du}(0) = 0$ *for each variation* Γ_u *of* Γ_0 *with fixed ends.*

This definition makes sense exactly because in local coordinates $q_1, \cdots, q_N$, if $\varpi = \sum_{i=1}^N \varpi_i(q) dq_i$ and if $[a, b] \mapsto q(t)$ is an oriented parametrization of Γ, the Lagrangian $L(q, \dot{q}) = \sum_{i=1}^N \varpi_i(q)\dot{q}_i$ is a *parametric* one, and hence the action is independent of the choice of the parametrization.

Lemma 2 *The curve* Γ *is stationary for* $\mathcal{A} = \int_\Gamma \varpi$ *if and only if it is tangent at each point to the kernel of* $d\varpi$*, that is if* Γ *is a* caracteristic curve *of* $d\varpi$*.*

Proof. Choose a parametrization $t \mapsto q(u,t)$ of a family Γ_u defined on an interval $[a,b]$ independent of u but with free endpoints. Let $t \mapsto X(t) = \frac{\partial q}{\partial u}(0,t)$ be the infinitesimal variation of $\Gamma_0 = \Gamma$. When applied to the Lagrangian $L(q,\dot{q}) = \sum_{i=1}^{N} \varpi_i(q)\dot{x}_i$ the formula giving the variation becomes

$$\frac{d\mathcal{A}}{du}(0) = \int_\Gamma i_X d\varpi \; + \; [\varpi(X)]_a^b .$$

It follows that a characteristic curve of $d\varpi$ is stationary among families of curves whose endpoints describe a curve tangent to the kernel of ϖ.

Now, if X is a vector field with flow φ_u and ϖ a differential 1-form on the manifold V, let us apply the above formula to the family $\Gamma_u = \varphi_u(\Gamma)$. One gets

$$\frac{d\mathcal{A}}{du}(0) = \frac{d}{du}_{|u=0}\left(\int_{\Gamma_u} \varpi\right) = \int_\Gamma \frac{d}{du}_{|u=0} \varphi_u^* \varpi = \int_\Gamma L_X \varpi,$$

where L_X is the *Lie derivative*. Finally, for any curve Γ,

$$\int_\Gamma L_X\varpi = \int_\Gamma i_X d\varpi + [\varpi(X)]_a^b = \int_\Gamma i_X d\varpi + d i_X \varpi,$$

which implies the *Cartan formula* for 1-forms:

$$L_X\varpi = i_X d\varpi + d i_X \varpi.$$

Replacing the curves by submanifolds dimension k, one proves the Cartan formula for differential k-forms.

1.3 Changing coordinates

Let $\varphi : M_1 \to M_2$ be a diffeomorphism and let $T\varphi : TM_1 \to TM_2$ be its *tangent map*, defined by

$$T\varphi(x,\dot{x}) = \big(\varphi(x), d\varphi(x)\dot{x}\big).$$

Definition 3 *Let the Lagrangians $L_1 : TM_1 \times \mathbb{R} \to \mathbb{R}$ and $L_2 : TM_2 \times \mathbb{R} \to \mathbb{R}$ be such that $L_1 = L_2 \circ (T\varphi \times Id)$. One says that L_1 is the inverse image of L_2 (resp. that L_2 is the direct image of L_1).*

The following fact shows how simple it is to change coordinates in the Euler-Lagrange equations:: *the Euler-Lagrange equations have exactly the same form in any (local) system of coordinates on M.* This will be superseded only by the Hamiltonian theory where configuration and velocity (more exactly momentum) coordinates will be on the same footing.

This assertion is obvious because if $c_1 : [a,b] \to M_1$ and $c_2 : [a,b] \to M_2$ are such that $c_2 = \varphi \circ c_1$, we have $L_1\big(c_1(t), \dot{c}_1(t), t\big) = L_2\big(c_2(t), \dot{c}_2(t), t\big)$, from which follows that c_1 is an extremal of L_1 if and only if c_2 is an extremal of L_2. (Here it was obviously more convenient to use the notation $t \mapsto c(t)$ than $t \mapsto q(t)$ for the path c)

More conceptually, if we define $[L]_c(t) \in T^*_{c(t)}M$ (the dual of $T_{c(t)}M$) by

$$[L]_c(t) := \frac{\partial L}{\partial q}\big(c(t), \dot c(t), t\big) - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot q}\big(c(t), \dot c(t), t\big)\right),$$

one can show that the mapping $[L]_c : [a, b] \to T^*M$ is a field of covectors tangent to M along the curve $c : [a, b] \to M$, which is intrinsically defined, independently of the choice of coordinates. This means that, with the above notations,

$$[L_1]_{c_1}(t) = [L_2]_{c_2}(t) \circ d\varphi\big(c_1(t)\big).$$

Finally, the derivative of the action takes the intrinsic (i.e. geometric) form

$$d\mathcal{A}_L(c)X = \int_a^b < [L]_c(t), X(t) > \, dt,$$

where for each t $<,>$ is the pairing between a vector $X(t)$ and a covector $[L]_c(t)$ tangent to M at $c(t)$.

1.4 The simplest example of a completely integrable system : the geodesic flow of a flat torus

The Lagangian $L : T^*\mathbb{T}^2 = \mathbb{R}^2/(2\pi\mathbb{Z})^2 \times \mathbb{R}^2 \to \mathbb{R}$ is $L(q, \dot q) = \frac{1}{2}||\dot q||^2$. We shall write the coordinates $q = (\varphi, \psi)$ and $\dot q = (\dot\varphi, \dot\psi)$ (figure 1.1).

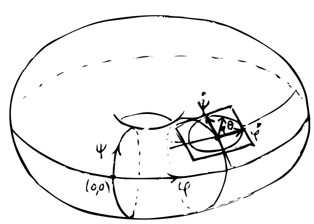


Figure 1.1

The Euler-Lagrange equation (E) is $\ddot q = 0$ and the extremals, the geodesics of $\mathbb{T}^2$, are the images by the canonical projection of the straight lines of $\mathbb{R}^2$ with an affine parametrization. The Legendre diffeomorphism is defined by $p = q$ and fixing the energy $H(p, q) = \frac{1}{2}||p||^2$ amounts to fixing the norm of the velocity. If the energy is different from 0, the energy hypersurface is diffeomorphic to $\mathbb{T}^3 = (\mathbb{R}/2\pi\mathbb{Z})^3$. The flow is depicted on figure 1.2.

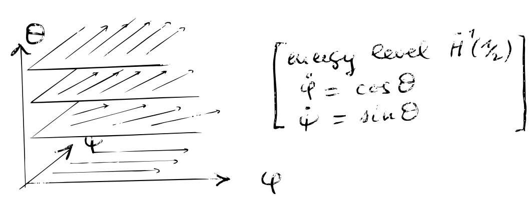


Figure 1.2

The whole *phase space* $T\mathbb{T}^2$ (or $T^*\mathbb{T}^2$) is foliated by the 2-dimensional tori $\dot q =$ constant (or $p =$ constant) which are invariant under the flow of X_L. On these tori, the vector field is constant (the flow is a flow of translations) and, depending on the rationality or irrationality of $\dot\psi/\dot\varphi$, the integral curves on the torus are all periodic or all dense.

Notice that the tori on which the integral curves are dense have a dynamical definition, as the closure of any of the integral curves they contain. This is not the case of the "periodic" tori which are a mere union of closed integral curves.

1.5 Opening of a resonance : the geodesic flow of a torus of revolution

We embed the 2-torus $\mathbb{T}^2 = \mathbb{R}^2/(2\pi\mathbb{Z})^2$ in $\mathbb{R}^3$ by the mapping $(r < 1)$

$$f : (\varphi, \psi) \mapsto \big((x = 1 + r\cos\psi)\cos\varphi, y = (1 + r\cos\psi)\sin\varphi, z = r\sin\psi\big).$$

The image is invariant under rotation around the z-axis. The Lagrangian is the kinetic energy, that is one half of the euclidean square length of the tangent vector $df(\varphi,\psi)(\dot\varphi,\dot\psi)$, that is

$$L(\varphi,\psi,\dot\varphi,\dot\psi) = \frac{1}{2}\left((1 + r\cos\psi)^2\dot\varphi^2 + r^2\dot\psi^2\right).$$

Its extremals are the geodesics of the induced metric. The conservation of energy says that they are parametrized with constant velocity, i.e. proportionnally to the arc length. As above, without loosing any geometric information, we fix the velocity to 1. The Euler-Lagrange equations are

$$\frac{d}{dt}\big(1 + r\cos\psi)^2\dot\varphi\big) = 0, \quad \frac{d}{dt}\big(r^2\dot\psi\big) = -r\sin\psi(1 + r\cos\psi)\dot\varphi^2.$$

The first expresses the invariance under rotation around Oz and can be interpreted as the *conservation of the angular momentum around Oz*. It is the analogue of the conservation of the angle θ in the flat case. Fixing the energy is fixing the velocity and the non-zero energy levels are diffeomorphic to the unit tangent bundle $T^1\mathbb{T}^2 \equiv \mathbb{T}^3$ with global angular coordinates (φ,ψ,θ) defined by choosing as third coordinate the *Riemannian angle* θ :

$$\dot\varphi = \frac{\cos\theta}{1 + r\cos\psi}, \quad \dot\psi = \frac{\sin\theta}{r}\ .$$

The first Euler-Lagrange equation becomes the constancy of the *Clairaut integral* :

$$(1 + r\cos\psi)\cos\theta = \text{ constant}.$$

Figures 1.4 represents the level curves of this function in the plane (ψ,θ). Figure 1.3 represents the level curves of the function θ, which plays for the flat torus the role of the Clairaut integral.

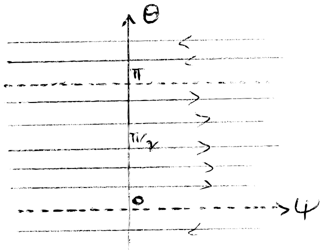


Figure 1.3

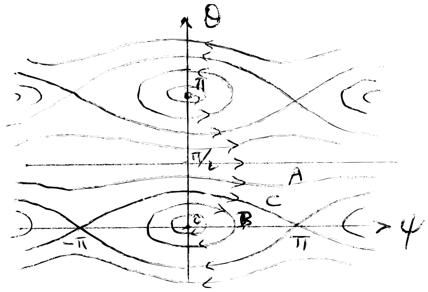


Figure 1.4

In the coordinates (φ, ψ, θ), the equations become

$$\frac{d\varphi}{dt} = \frac{\cos\theta}{1 + r\cos\psi}, \quad \frac{d\psi}{dt} = \frac{\sin\theta}{r}, \quad \frac{d\theta}{dt} = \frac{-\cos\theta\sin\psi}{1 + r\cos\psi}.$$

Because of the invariance under rotation, they are independent of φ, hence they admit a direct image in the torus (ψ, θ) which consists in ignoring the first equation. The same is of course true for the flat metric. The integral curves of this direct image are contained in the level curves of the Clairaut integral, which explains the arrows of figures 1.3 and 1.4.

In each open band $\theta \in]-\frac{\pi}{2}+k\pi, \frac{\pi}{2}+k\pi[$, $k \in \mathbb{Z}$, the flow looks qualitatively like the flow of a conservative pendulum. The rotations of the pendulum correspond to integral curves belonging to invariant tori which, as in the flat case, project biunivocally onto the configuration torus (φ, θ) (type A, figure 1.5),

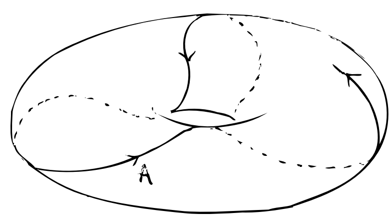


Figure 1.5

while oscillations correspond to integral curves belonging to invariant tori which project neither injectively nor surjectively but on an annulus whose boundary is a *caustic* (type B, figure 1.6). The latter tori fill the *resonance zone.*

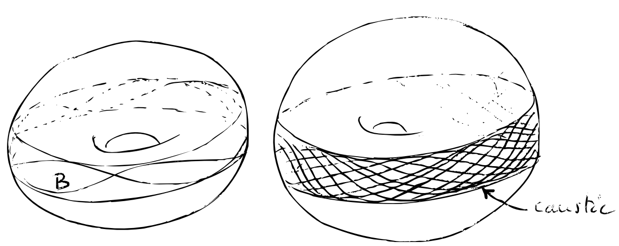


Figure 1.6

As in the flat case, in each of these invariant tori, integral curves are either all periodic or all dense. A new feature is the existence in each non-zero energy level of 4 isolated periodic solutions, which correspond to the 2 geodesics defined by the intersection of the torus with the plane $z = 0$, each one with two possible directions of velocity. The inner one ($\psi = \pi, \theta = 0$ or $\psi = \pi, \theta = \pi$) is *hyperbolic* hence *unstable*. The set of integral curves with the same energy which are positively (negatively) asymptotic to it define the *stable* (*unstable*) manifold of this periodic orbit. These sets happen to coincide here. A corresponding geodesic (type C) is represented on figure 1.7. Their union is a surface which makes the transition between the two kinds of invariant tori oustside and inside the resonance zone. The outer one ($\psi = 0, \theta = 0$ or $\psi = 0, \theta = \pi$) is *elliptic*, hence stable. In its energy level, it is "surrounded" by invariant tori.

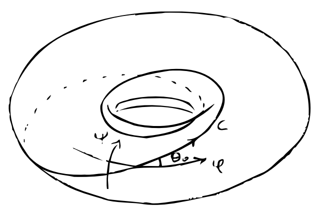


Figure 1.7

We have now two kinds of invariant sets dynamically defined : the invariant tori with dense integral curves and the stable = unstable manifolds of the hyperbolic periodic solutions.

A geometric interlude. Let us consider a type B geodesic close enough to the outside elliptic periodic geodesic (one chooses an orientation on it) around which it oscillates, intersecting it an infinite number of times. We shall show that the time between two successive intersections, i.e. the length of the corresponding geodesic segment, tends to $\pi\sqrt{r(1+r)}$ when the intersection angle θ at the initial instant tends to 0 (figure 1.8, left; in the flat case, there is no oscillation and the time between two intersections tends to ∞ when θ tends to 0, see figure 1.8, right).

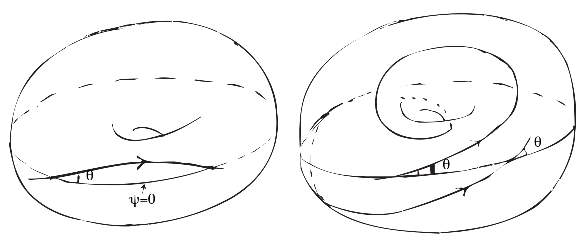


Figure 1.8

Here is a sketch of proof: close to the closed curve $\psi = \theta = 0$, the geodesic flow

$$\frac{d\varphi}{dt} = \frac{\cos\theta}{1 + r\cos\psi}\,, \quad \frac{d\psi}{dt} = \frac{\sin\theta}{r}\,, \quad \frac{d\theta}{dt} = \frac{-\cos\theta\,\sin\psi}{1 + r\cos\psi}\,,$$

may be replaced by the flow of its linearization along this curve, obtained by replacing the vector field by its Taylor expansion at order 1 in the variables (ψ, θ) at the point $(0,0)$:

$$\frac{d\varphi}{dt} = \frac{1}{1+r}, \quad \frac{d\psi}{dt} = \frac{\theta}{r}, \quad \frac{d\theta}{dt} = \frac{-\psi}{1+r}.$$

In such a neighborhood, a type B invariant torus is replaced by an invariant otrus of the linearized equation, that is a level hypersurface of the function $1 + r - r\frac{\psi^2}{2} - (1+r)\frac{\theta^2}{2}$, which is the Taylor expansion of order 2 at $(0,0)$ of the Clairaut integral $(1 + r\cos\psi)\cos\theta$. When the initial angle θ tends to 0, the time between two intersections of a type B geodesic and the elliptic periodic geodesic tends to half the period of the solutions of the linear equation

$$\frac{d\psi}{dt} = \frac{\theta}{r}, \quad \frac{d\theta}{dt} = \frac{-\psi}{1+r}.$$

This time can be readily computed from the eigenvalues $\pm\frac{i}{\sqrt{r(1+r)}}$ of the matrix of the equation: it is equal to $\tau = \pi\sqrt{r(1+r)}$, which is also the limit when θ tends to 0 of the distance between two consecutive intersections. If the torus of revolution is replaced by a round sphere of radius ρ in $\mathbb{R}^3$, the géodesics are great circles and intersect two by two in antipodal points; the length of a half grat circle is $\cdots \pi\rho = \pi\sqrt{\rho \cdot \rho}$, a formula very similar to the above one.

Indeed, $\frac{1}{r(1+r)}$ is the Gauss curvature of the torus at any point of its elliptic geodesic. With our little computation we have proved a very remarquable result: the a priori *extrinsic* (i.e.defined by the embedding in $\mathbb{R}^3$ of the torus) quantity $\tau = \pi\sqrt{r(1+r)}$ depends only on the *intrinsic* geometry of the torus (i.e. of its riemannian metric). That is, we have proved, in this very particular case, Gauss' *Theorema egregium* (i.e. the wonderful theorem).

1.6 Straightening the flow on the invariant tori

The invariant tori $\mathcal{T}_{\theta_0}$ of the geodesic flow of the flat torus $\mathbb{T}^2$ are defined by the equations $\theta = \theta_0$; hence, they project bijectively on the geometric torus $\mathbb{T}^2$. In the coordinates (φ, ψ), the restriction Φ_t to $\mathcal{T}_{\theta_0}$ of the geodesic flow is affine:

$$\Phi_t(\varphi, \psi) = (\varphi + t\cos\theta_0, \psi + t\sin\theta_0).$$

The existence on an invariant torus of global angular coordinates such the restriction of the geodesic flow is affine is a very general property of the so-called *completely integrable hamiltonian systems.* We shall show now that this property is shared by the invariant tori of the geodesic flow of a torus of revolution. The proof is essentially contained in figure 9, the important thing being the invariance under translations (rotations) $\varphi \mapsto \varphi + \varphi_0$ of the time t_0 necessary for an integral curve starting on the circle $\mathcal{S}$ defined by $\eta = \eta_0$ to come back to it (η being an angular coordinate on a level circle of the Clairaut integral in the torus (ψ, θ)).

One defines a diffeomorphism F of the torus (φ, η) onto itself by the formula

$$F(\varphi, \eta) = \left(\varphi_0(\varphi, \eta) + \omega \frac{t(\eta)}{t_0} \,,\ \eta_0 + 2\pi \frac{t(\eta)}{t_0} \right),$$

where ω is the angle of the rotation P, which is the first return map of the flow on the circle $\mathcal{S}$, $\big(\varphi_0(\varphi,\eta), \eta_0\big)$ is the first intersection point with $\mathcal{S}$ of the integral curve through (φ, η) described in the negative direction (see figure 9), $t(\varphi, \eta) = t(\eta)$ is the time taken by the flow between these two points, that is $\Phi_{t(\eta)}\big(\varphi_0(\varphi,\eta), \eta_0\big) = (\varphi, \eta)$ (where Φ_t is the flow), and t_0 is the return time on $\mathcal{S}$.

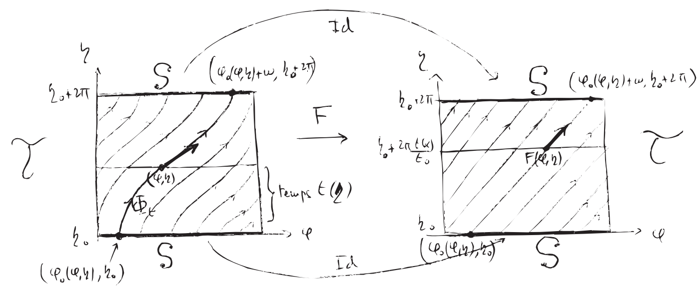


Figure 1.9

The difféomorphism F transforms le vector field on $\mathcal{T}$ into the constant vector field $\left(\frac{\omega}{t_0}, \frac{2\pi}{t_0}\right)$, and hence the flow into the affine flow

$$(\varphi, \eta) \mapsto \left(\varphi + \omega \frac{t}{t_0} \,,\ \eta + 2\pi \frac{t}{t_0} \right).$$

2 Integral invariants and Hamilton's equations

2.1 The Poincaré-Cartan integral invariant

From the fundamental formula, it follows that, if γ_u is a family of extremals of the action $\mathcal{A}_L = \int L\,dt$, we get

$$\frac{d\mathcal{A}_L}{du} = \left[\frac{\partial L}{\partial \dot q}\cdot\frac{\partial q}{\partial u} + L\,\frac{dt}{du}\right]_a^b .$$

We replace now the partial derivative $\frac{\partial q}{\partial u}$ (that is $\frac{\partial \Gamma}{\partial u}$), deprived of geometric meaning, by the "effective variation"

$$\frac{d}{du}\Big(q\big(u,t(u)\big)\Big) := \frac{dq}{du} = \frac{\partial q}{\partial u} + \frac{\partial q}{\partial t}\frac{dt}{du} = \frac{\partial q}{\partial u} + \dot q\,\frac{dt}{du}, \quad t(u) = a(u) \text{ or } b(u),$$

of the extremities of the path γ_u as a fonction of u (figure 2.1).

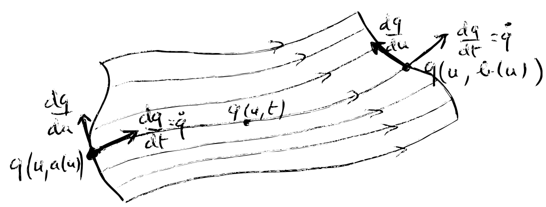


Figure 2.1

This transforms the expression of $\frac{d}{du}\left(\mathcal{A}_L(\gamma_u)\right)$ for a family of extremals into an identity between *differential 1-forms* on the interval $\mathcal{U}$ of definition of the parameter u :

$$d\mathcal{A}_L = \delta_b^*\varpi_L - \delta_a^*\varpi_L,$$

where $\delta_a, \delta_b : \mathcal{U} \to T^*M \times \mathbb{R}$ denote the mappings

$$\delta_t(u) = \left(q\big(u,t(u)\big), \frac{\partial q}{\partial t}\big(u,t(u)\big), t(u)\right), \quad t(u) = a(u) \text{ or } b(u),$$

and ϖ_L is the differential 1-form on $TM \times \mathbb{R}$ defined by

$$\varpi_L = \frac{\partial L}{\partial \dot q}(q,\dot q,t)\cdot dq - \left(\frac{\partial L}{\partial \dot q}(q,\dot q,t)\cdot \dot q - L(q,\dot q,t)\right)dt.$$

Finally, we can simplify the formulas by transporting everything on the cotangent side with the Legendre diffeomorphism $\Lambda : (q,\dot q,t) \mapsto (q,p,t)$. The function on $T^*M \times \mathbb{R}$ defined by

$$H(q,p,t) = p\cdot\dot q - L(q,\dot q,t),$$

where $\dot q$ is expressed in terms of p,q,t via Λ is called the *Legendre transform* of L, or the *Hamiltonian* associated to the Lagrangian L. We have already met this function under the name of "total energy".

If ϖ_H denotes the 1-form on $T^*M \times \mathbb{R}$ defined by

$$\varpi_H = p \cdot dq - H(q, p, t)dt,$$

the formula for the unconstrained variations of extremals becomes

$$d\mathcal{A}_L = (\Lambda \circ \delta_b)^* \varpi_H - (\Lambda \circ \delta_a)^* \varpi_H.$$

The 1-form ϖ_H is the *Poincaré-Cartan integral invariant* (*tenseur impulsion-énergie* in Cartan's terminology).

Rewriting the action. As $L = p \cdot \dot{q} - H$, the action istself can now be written as the integral of $\varpi_H = p \cdot dq - Hdt$ on the *lift* $\Gamma^*(t) = \Big(q(t), \frac{\partial L}{\partial \dot{q}}(q(t), \dot{q}(t), t), t\Big)$ to $T^*M \times \mathbb{R}$ of the path $\gamma(t)$ in M :

$$\mathcal{A}_L(\gamma) = \int_{\Gamma^*} \varpi_H.$$

This expression is the basis of the *least action principles of Hamilton and Maupertuis* (see theorems 9 and 11).

2.2 The symplectic structure and Hamilton's equations

Euler-Lagrange equations are equivalent to saying that a path $t \mapsto q(t)$ in M is an extremal if and only if the parametrized curve in $T^*M \times \mathbb{R}$

$$t \mapsto \left(q(t), \frac{\partial L}{\partial \dot{q}}\big(q(t), \dot{q}(t), t\big), t\right) = \Lambda\big(q(t), \dot{q}(t), t\big)$$

is an integral curve of the vector field

$$\Xi_H^* = (X_H^*, 1) = \Lambda_*(X_L, 1)$$

on $T^*M \times \mathbb{R}$. The last formula of the preceding section then implies that, if C_a and C_b are two oriented *loops* in $T^*M \times \mathbb{R}$, such that $C_b - C_a$ is the oriented boundary of a cylinder C generated by pieces of of integral curves of $\Xi_H^* = (X_H^*, 1)$, one has

$$\int_{C_a} p \cdot dq - H(q, p, t)dt = \int_{C_b} p \cdot dq - H(q, p, t)dt\,.$$

Definition 4 (Integral invariant) *Let V be a manifold, Ξ a vector field on V. One says that the 1-form ϖ is an integal invariant of Ξ if the equality $\int_{C_0} \varpi = \int_{C_1} \varpi$ holds whatever be the couple of oriented loops C_0, C_1 such that $C_0 - C_1$ be the oriented boundary of a cylinder C made of integral segments of the vector field Ξ.*

Hence, the Poincaré-Cartan 1-form $\varpi_H = p \cdot dq - Hdt$ is an integral invariant of the vector field $\Xi_H^* = (X_H^*, 1)$ on $V = T^*M \times \mathbb{R}$. In E. Cartan's terminology, what we have called an integral invariant is a (*relative* and *complete*) integral invariant: relative because its invariance holds only if the integral is taken on loops C_i, complete because C_0 and C_1 are not supposed to be the image of each other under the flow of Ξ (i.e. the time to go from one to the other is not the same for all segments).

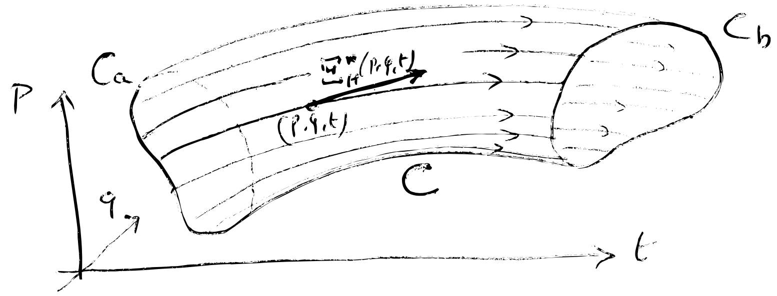


Figure 2.2

However, an important property of X^*_H comes from applying Stokes formula to small disks D_a et D_b contained respectively in the time slices $T^*M \times \{a\}$ and $T^*M \times \{b\}$ and such that $D_b = \varphi^b_a(D_a)$ is the image of D_a under the flow of Ξ^*_H. Indeed, one gets the

Theorem 5 *The time-dependent vector field X^*_H defined on T^*M, preserves the* standard symplectic 2-form $\omega = \sum_{i=1}^n dp_i \wedge dq_i$.

A corollary of the preservation of the symplectic structure is

Theorem 6 (Liouville's theorem) *The flow of the time-dependent vector field X^*_H preserves the 2n-form ω^n, hence the Lebesgue measure (volume).*

Hamilton's equations. We now deduce the structure of the vector field X^*_H (i.e. the structure of the Euler-Lagrange equations (E) seen from the cotangent side) from the following characterization of integral invariants:

Lemma 7 (Infinitesimal characterization of integral invariants) *The 1-form ϖ on the manifold V is an integral invariant of the vector field Ξ if and only if, at each point $v \in V$ the vector $\Xi(v)$ belongs to the kernel of the bilinear form $d\varpi(v)$, i.e. if $i_\Xi d\varpi = 0$.*

Proof. One may suppose that Ξ does not vanish. Let C be an embedded cylinder made of segments of integral curves of Ξ, and $C_1 - C_0$ the oriented boundary of C. We suppose that C_0 and C_1 bound oriented embedded disks, respectively D_0 and D_1, where D_1 is obtained from D_0 by following the integral curves of Ξ (of course, the time is not supposed to be constant). The formal difference $C_1 - C_0$ is also the oriented boundary of $D_1 - D_0$ (figure 2.3). Two these two possibilities correspond two proofs of the lemma, both based on Stokes formula.

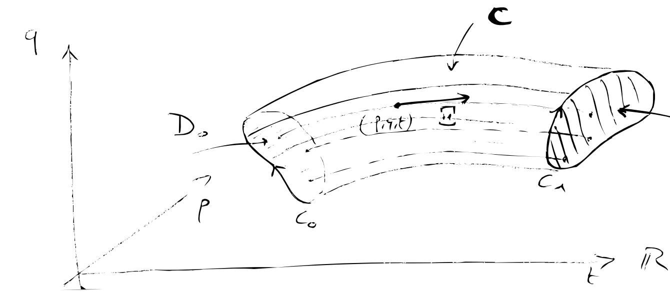


Figure 2.3

In the first one, one writes

$$0 = \int_{C_1} \varpi - \int_{C_0} \varpi = \int_C d\varpi.$$

Being true for all cylinders C made of segments of integral curves, it implies the existence of a function F which to each couple (I_0, I_1) of integral segments of Ξ associates the common value of the integral of $d\varpi$ on any oriented"rectangle" $\mathbb{R}$, which is the union of a family of integral segments joining I_0 to I_1 (figure 2.4(i)). It remains to notice that this function is necessarily identically equal to 0 (figure 2.4(ii)).

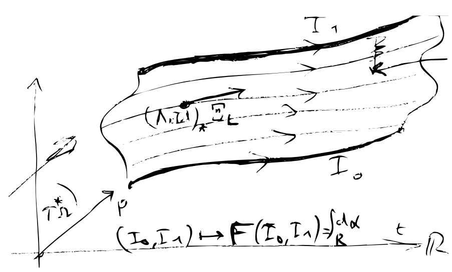


Figure 2.4(i)

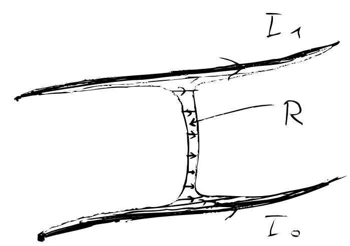


Figure 2.4(ii)

Finally, considering two integral segments I_0 et I_1 which are infinitely close (or by introducing adapted coordinates on a rectangle), one deduces that for any point $v \in V$, and any tangent vector ξ at this point,

$$d\varpi(v)(\Xi(v), \xi) = 0,$$

which is the assertion of the lemma.

In the second proof, one writes

$$0 = \int_{C_1} \varpi - \int_{C_0} \varpi = \int_{D_1} d\varpi - \int_{D_0} d\varpi.$$

This implies the preservation of the 2-form $d\varpi$ by any vector field on V whose integral curves coincide with those of Ξ, that is any vector field of the form $f\Xi$, where f is a differentiable function on V which does not vanish. Infinitesimally, this is equivalent to

$$\forall f, \quad L_{f\Xi} d\varpi = 0,$$

where $L_X\omega = i_X d\omega + d i_X \omega$ is the Lie derivative of ω along X (see section 7.7), hence

$$\forall f, \quad L_{f\Xi} d\varpi = d i_{f\Xi} d\varpi = f(d i_\Xi d\varpi) + df \wedge (i_\Xi d\varpi) = 0.$$

If f is constant, $df = 0$ hence $d i_\Xi d\varpi = 0$. But then, for any f, $df \wedge (i_\Xi d\varpi) = 0$, from which follows that $i_\Xi d\varpi = 0$, which is the conclusion of the lemma

Applying this to the integral invariant $p \cdot dq - H dt$ determines the direction of Ξ_H^*, hence X_H^*, because the kernel of

$$d(p \cdot dq - H dt) = \sum_{i=1}^{n} \left[\left(dp_i + \frac{\partial H}{\partial q_i} dt \right) \wedge \left(dq_i - \frac{\partial H}{\partial p_i} dt \right) \right],$$

is easily seen to be 1-dimensional and generated at each point (p, q, t) by the vector $\left(-\frac{\partial H}{\partial q}(q,p,t), \frac{\partial H}{\partial p}(q,p,t), 1 \right)$.

Finally, we get

$$X_H^* = \left(-\frac{\partial H}{\partial q_1}, \cdots - \frac{\partial H}{\partial q_n}, \frac{\partial H}{\partial p_1}, \cdots \frac{\partial H}{\partial p_n} \right).$$

Hence, when transported on the cotangent side by the Legendre diffeomorphism, Euler-Lagrange equations (E) take the particularly symmetric form of *Hamilton's equations* (or *canonical equations*) :

$$\frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}, \ \frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}, \quad i = 1 \cdots n.$$

As the equations depend only on H, this justifies the notation X_H^*. It is fair to remember that this particularly symmetric form of the equations of classical mechanics already appears in Lagrange's works.

Exercise. Notice that Hamilton's equations are equivalent to the identity

$$i_{X_H^*} \omega = -\partial H, \quad \text{where} \quad \partial H = \frac{\partial H}{\partial p} \cdot dp + \frac{\partial H}{\partial q} \cdot dq = dH - \frac{\partial H}{\partial t} dt.$$

Alternatively, deduce this directly from the identity $i_{\Xi_H^*} d\varpi_H = 0$.

The example of classical mechanics. The Lagrangian is the difference

$$L(q, \dot q) = \frac{1}{2} \dot q \cdot G(q) \dot q - V(q) = \frac{1}{2} g(q)(\dot q, \dot q) - V(q)$$

between *kinetic* and *potential* energy. The kinetic energy is defined by a Riemannian metric g on M, that is for each q a positive definite quadratic form $g(q)$, represented by a symmetric matrix $G(q)$. When there is no potential V, the extremals are the *geodesics* of the metric. The Legendre transform $p = G(q)\dot q$

defines the conjugate momenta (the *impulsions*) p_i of the configuration variables q_i, the $\dfrac{\partial L}{\partial q_i}$ are the *forces* and the Hamiltonian is total energy, i.e. the sum of kinetic and potential energies

$$H(q,p) = \frac{1}{2}\,\dot q \cdot G(q)\dot q + V(q) = \frac{1}{2} p \cdot G(q)^{-1} p + V(q).$$

Legendre transform in the convex case: Young-Fenchel inequality. It follows from Hamilton's equations that the Legendre transform $L \mapsto H$ is involutive :

$$H(q,p,t) = p\cdot \dot q - L(q,\dot q,t), \quad p = \frac{\partial L}{\partial \dot q}(q,\dot q,t),$$

$$L(q,\dot q,t) = p\cdot \dot q - H(q,p,t), \quad \dot q = \frac{\partial H}{\partial p}(q,p,t).$$

This symmetry makes it natural to write the correspondance $L \leftrightarrow H$ in the following form, where the variables (q,t) play the role of mere parameters :

$$p\cdot \dot q = L(q,\dot q,t) + H(q,p,t).$$

The convexity of $\dot q \mapsto L(q,\dot q,t)$ is equivalent to that of $p \mapsto H(q,p,t)$ and if a function satisfies the general convexity hypotheses, so does its transform.

Theorem 8 (Young-Fenchel inequality) *For all $q,t,\dot q,p$, the following holds :*

$$p\cdot \dot q \le L(q,\dot q,t) + H(q,p,t),$$

with equality if and only if $p = \dfrac{\partial L}{\partial \dot q}(q,\dot q,t)$.

Figure 2.5 illustrates in dimension 1 this variational definition of the Legendre transform. One also reads on this figure the interpretation of the transform as the passage from a punctual to a tangential equation.

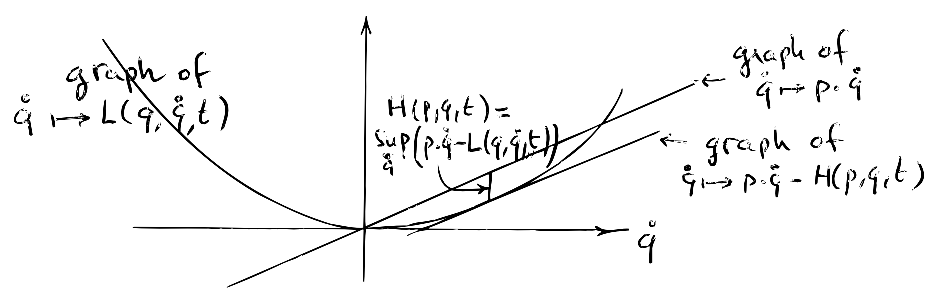


Figure 2.5

Finally, from Lemmas 2 and 7 one deduces the

Theorem 9 (Hamilton's variational principle) *Given a Hamiltonian $H : T^*M \times \mathbb{R} \to \mathbb{R}$, the (unparametrized) integral curves of the vector field Ξ^*_H, i.e. the graphs of solutions of X^*_H, are the extremals of the Hamilton's action*

$$\mathcal{H} = \int_\Gamma \varpi^*_H$$

within the set of oriented 1-dimensional submanifols Γ of $T^*M \times \mathbb{R}$ which join the subspaces $\pi^{-1}(q_0)\times\{t_0\}$ and $\pi^{-1}(q_1)\times\{t_1\}$, where π is the natural projection of T^*M on M (i.e. the projection $(p,q) \mapsto q$).

Remarks. 1) One can show directly the corollary by writing Euler-Lagrange equations of the Lagrangien $\mathcal{L}$ sur $T(T^*\Omega) \times \mathbb{R}$ defined by

$$\mathcal{L}(p,q,\dot p,\dot q,t) = p\cdot\dot q - H(p,q,t).$$

2) When $H = p\cdot\dot q - L$ is the Legendre transform of a regular Lagrangian L, the integral $\int_\Gamma p\cdot dq - Hdt$ on the graph Γ of the solution $[a,b] \ni t \to \big(p(t),q(t)\big)$ of Hamilton's equations coincide with the action $\int_a^b Ldt$ of the solution $[a,b] \ni t \to q(t)$ of the Euler-Lagrange equations of L. Hence the extremals of both integrals are the same, in spite of the fact that the space of paths at stake in Hamilton's principle is much larger than the set of paths in Ω (no a priori relation is assumed betwee $p(t)$ and $q(t)$); the explanation is that, $q,\dot q,t$ being given, $p\cdot\dot q - H$ is an extremum implies $\dot q = \dfrac{\partial H}{\partial p}$, i.e. $p = \dfrac{\partial L}{\partial \dot q}$.

2.3 Time and energy as conjugate variables.

Let $T^*M \to \mathbb{R}$ be an autonomous (i.e. independent of time) Hamiltonian. As $dH\cdot X^*_H = -\omega(X^*_H, X^*_H) = 0$, which in coordinates becomes

$$d/dt\left(H\big(p(t),q(t)\big)\right) = -\dot q\cdot\dot p + \dot p\cdot\dot q = 0,$$

the vector field X^*_H defined by H on T^*M leaves the function H invariant. This is the *energy conservation* we have already met in the first chapter when looking at the internal variations. Hence the integral curves of X^*_H are contained in the level hypersurfaces of H

$$\Sigma_h = H^{-1}(h) = \{\alpha \in T^*M,\ H(\alpha) = h\}.$$

If h is a *regular value*[2] of H, we shall denote by $X^*_{H,h}$ the restriction of X^*_H to Σ_h. Note that in this case $\operatorname{grad} H$, and hence also X^*_H, does not vanish on Σ_h. Consider the *Liouville 1-form* λ which in coordinates is $\lambda = p\cdot dq = \sum_{i=1}^n p_i dq_i$, and let i_h be the canonical injection of Σ_h in T^*M. The 1-form $\lambda_h = i_h^*\lambda$ induced by λ on Σ_h plays with respect to $X^*_{H,h}$ the part played by ϖ^*_H with respect to $\Xi^*_H = (X^*_H, 1)$ (compare the following Lemma to Lemma 7):

[2] i.e. if the derivative dH does not vanish anywhere on Σ_h which, thanks to the implicit function theorem, implies that Σ_h is a $2n-1$-dimensional submanifold of T^*M. Sard's theorem asserts that "almost every" energy level is regular.

Lemma 10 *In a regular level Σ_h of H, the vector field $X^*_{H,h}$ generates the field of kernels of the 2-form $\omega_h = i_h^*\omega = d\lambda_h$.*

Proof. Let $\alpha \in \Sigma_h$; the kernel of $\omega_h(\alpha)$ is the set of X in $T_\alpha\Sigma_h$ such that

$$\forall Y \in T_\alpha\Sigma_h, \omega_h(\alpha)(X,Y) = 0.$$

Hence it is the set of $X \in T_\alpha\Sigma_h$ such that

$$dH(\alpha)(Y) = 0 \Longrightarrow \omega(\alpha)(X,Y) = 0.$$

The regularity of Σ_h and the non degeneracy of ω imply that, when considered as equations in $T_\alpha T^*M$, each side of the implication defines a hyperplane. Hence the implication is equivalent to the existence of a non zero constant k such that $i_X\omega(\alpha) + kdH(\alpha) = 0$. But, as H is autonomous, we have $i_{X^*_H}\omega = -dH$. Hence the vector $X - kX^*_{H,h}(\alpha)$, which belongs to the kernel of the non degenerate 2-form $\omega(\alpha)$, is equal to 0.

In the same way as, thanks to lemma 2, the property that Ξ^*_H generates the kernel of $d\varpi^*_H$ is equivalent to the *Hamilton variational principle* (theorem 9), which holds as well for non autonomous Hamiltonians, lemma 10 is equivalent to a variational principle which holds only for autonomous Hamiltonians:

Theorem 11 (Maupertuis' variational principle) *Let $H(p,q)$ be a time independent Hamiltonian. The solutions of Hamilton's equations which, possibly after reparametriation, are contained in a regular energy level Σ_h, are the extremals of the integral $\mathcal{M}_h(\Gamma) = \int_\Gamma \lambda_h$ within the set of 1-dimensional oriented submanifolds of Σ_h joining two subspaces of the form $\Sigma_h \cap \{q = q_0\}$ and $\Sigma_h \cap \{q = q_1\}$).*

Finally, one deduces from lemma 7 that λ_h is an integral invariant of $X^*_{H,h}$, that is:

Theorem 12 (Poincaré integral invariant, fixed energy version) *If C_0 and C_1 are two oriented loops in Σ_h such that $C_1 - C_0$ is the oriented boundary of a cylinder C formed by integral segments of $X^*_{H,h}$, one has*

$$\int_{C_0} \lambda_h = \int_{C_1} \lambda_h.$$

Moreover, the direction (but not the length) of $X^*_{H,h}$ is completely determined by this property .

Of course, the conclusion of Theorem 12 still holds, even for a non autonomous Hamiltonian, if instead of fixing the energy we fix the time:

Theorem 13 (Poincaré integral invariant, fixed time version) *Let $H : T^*M \times \mathbb{R} \to \mathbb{R}$ be a Hamiltonian which may depend on time. If C_0 and C_1 are* *two oriented loops in T^*M such that C_1 be the image of C_0 under the flow $\varphi_{t_0}^{t_1}$ of X_H^* for some couple of times (t_0, t_1), one has*

$$\int_{C_0} \lambda = \int_{C_1} \lambda,$$

where λ is the Liouville canonical 1-form on T^*M.

This duality between time and energy is particularly in evidence on the identity

$$d(p \cdot dq - H(p,q,t)dt) = d(p \cdot dq + tdH(p,q,t)).$$

Back to geodesics: the Jacobi metric. One of the best known applications of the "elimination of time" effected by the Maupertuis principle is the reduction, due to Jacobi, of problems of classical mechanics to problems of geodesics. This is the first apparition, (true, with fixed energy), of the possibility to replace forces by geometric properties of space, an idea which will lead to *general relativity*. Let (M,g) be a Riemannian manifold and $L : TM \to \mathbb{R}$ a *classical Lagrangian*

$$L(\xi) = \frac{1}{2}||\xi||_g^2 - V(\pi(\xi)), \quad (\pi : TM \to M \quad \text{canonical projection}),$$

to which the Legendre transform associates (exercise !) the Hamiltonien

$$H(\alpha) = \frac{1}{2}||\alpha||_{g^{-1}}^2 + V(\tilde{\pi}(\alpha)), \quad (\tilde{\pi} : T^*M \to M, \text{ canonical projectione}),$$

In local coordinates,

$$L(q,\dot{q}) = \frac{1}{2}\dot{q} \cdot G(q)\dot{q} - V(q) \quad \text{et} \quad H(p,q) = \frac{1}{2}p \cdot G^{-1}(q)p + V(q).$$

Leit h be a regular value of H; one deduces from Maupertuis 11 that those extremals $c(t)$ of $\int L dt$ whose energy is h, i.e. those which satisfy

$$||\dot{c}(t)||_{g(c(t))}^2 = 2\big(h - V(c(t))\big), \qquad (\star_h)$$

are, up to reparametrization, the extremals of $\int ||\dot{c}(t)||_{g(c(t))}^2\, dt$ within the set of parametrized curves $t \mapsto c(t)$ in M joining two given points m_0 and m_1 and satisfying $(\star_h)$. Thanks to remark 2 at the end of section 2.2, one can indeed restrict to curves in Σ_h which are images by the Legendre transformation of curves of the form $(c(t), \dot{c}(t))$.

Theorem 14 *The extremals with energy h of a classical Lagrangian $L(q,\dot{q}) = \frac{1}{2}||\dot{q}||_g^2 - V(q)$ may be defined, up to parametrization, as the extremals of the length integral $\int ||\dot{q}||_{\tilde{g}}\, dt$ for the Riemannian metric $\tilde{g}$ which is defined on $M_h = \{m \in M,\ h - V(m) > 0\}$ by $\tilde{g}(m) = 2\big(h - V(m)\big)\, g(m)$.*

Proof. It is enough to notice that, on the set of curves which satisfy $(\star_h)$, one has

$$\int ||\dot c(t)||^2_{g(c(t))}\,dt = \int ||\dot c(t)||_{\tilde g(c(t))}\,dt,$$

and that the integral on the right depends only on the image and not on the parametrization $t \mapsto c(t)$ of the curve. This allows forgetting the initial constraint on parametrization.

Remarques. 1) When the potential V vanishes and $h > 0$, the metric $\tilde g$ coincides with g up to a multiplicative constant; this proves that the geodesics of a metric are, up to parametrization, the extremals of the length.

2) The domains M_h to which the extremals with energy h are confined, are called *Hill's regions* (see figure 6.5) by reference to the pionneering works of George William Hill on the restricted 3-body problem at the end of 19^{th} century.

2.4 From time-dependent to time-independent: extension of the phase space

When proving that the conservation of energy for time autonomous Lagrangians is a consequence of the extremality under the sole internal variations (i.e. changes of the time parametrization), we had a first glance at the duality between time and energy. This duality was also at stake in theorems 12 and 13. We now make this idea more precise: a time-dependent system can always be embedded into a time-independent one at the expense of adding dimensions and loosing track of time origin. Indeed, the vector field $X_K^* = \left(-\frac{\partial H}{\partial q}, -\frac{\partial H}{\partial t}, \frac{\partial H}{\partial p}, 1\right)$ on $T^*(M\times\mathbb{R})$ corresponding to the *extended* Hamiltonian

$$K(p,E,q,\tau) = E + H(p,q,\tau)$$

restricts to $\Xi_H^* = \left(-\frac{\partial H}{\partial q}, \frac{\partial H}{\partial p}, 1\right)$ when identifying with $T^*M\times\mathbb{R}$, by the projection forgetting E, the energy hypersurface $K^{-1}(0) \subset T^*(M\times\mathbb{R}) \equiv T^*M\times\mathbb{R}^2$ (see figure 2.6).

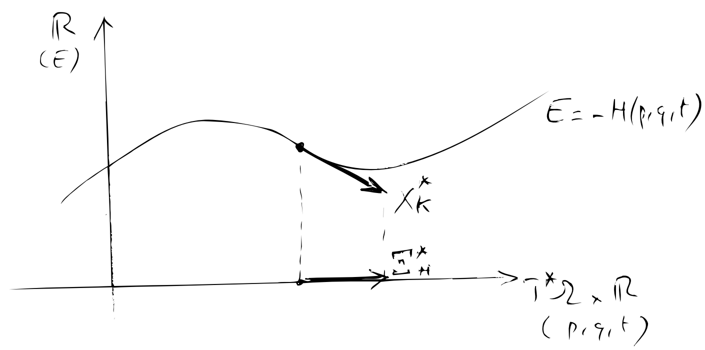


Figure 2.6

In the same way, the *Liouville form* $p\cdot dq + E\cdot dt$ on $T^*(M\times\mathbb{R})$ becomes the Poincaré Cartan integral invariant ϖ_H on $T^*M\times\mathbb{R}$.

This extension may be useful even if H does not depend on time: as we shall stress in the next section, the geometry of the energy hypersurface $K^{-1}(0)$ determines the direction of the restriction of the vector field X_K, that is of $\Xi_H^* = (X_H^*, 1)$; hence it determines *completely* the vector field X_H^*. Note however that K does not satisfy the Legendre condition.

Conversely, suppose that an autonomous Hamiltonian $H(p, q)$ satisfies $\frac{\partial H}{\partial p_n} \neq 0$ at some point of Σ_h. The implicit function theorem asserts that, in a neighborhood of such a point, Σ_h admits an equation of the form

$$p_n + K(\tilde{p}, \tilde{q}, q_n) = 0,$$

where $(\tilde{p}, \tilde{q}) = (p_1, \cdots, p_{n-1}, q_1, \cdots, q_{n-1})$. Hence, locally, one can consider Σ_h as the product by $\mathbb{R}$ of the "reduced" phase space $(\tilde{p}, \tilde{q})$ and $K(\tilde{p}, \tilde{q}, q_n)$ as a Hamiltonian on this space depending on the new time q_n. This gives a direct relation between the variational principles. of Hamilton and Maupertuis.

Exercise. Compare the flow of this non autonomous Hamiltonian with the projection on the space $(\tilde{p}, \tilde{q})$ of the flow of the restriction $X_{H,h}^*$ of X_H^* to Σ_h. Note that, as $\frac{dq_n}{dt} = \frac{\partial H}{\partial p_n} \neq 0$, the two times are comparable.

3 Symplectic manifolds and Hamiltonian vector fields

3.1 Definition and examples

Definition 15 *Let V be a manifold (necessarily of even dimension $2n$). A* symplectic form *on V is a differential 2-form ω which is closed ($d\omega = 0$) and regular (the biilinear form on T_zV it defines at each point z of V is non-degenerate, which means that the correspondence $X \mapsto (Y \mapsto \omega(X, Y))$ is an isomorphism from T_zV onto T_z^*V). The pair (V, ω) is called a* symplectic manifold. *If moreover the 2-form ω is a coboundary, that is if there exists a (Liouville) 1-form λ such that $\omega = d\lambda$, one says that (V, ω) is an exact symplectic manifold.*

The typical example is the standard symplectic form $\omega = d\lambda = dp \wedge dq$ on a cotangent space T^*M, where $\lambda = p \cdot dq$ is the Liouville form (which can be defined intrinsically). Darboux' theorem aserts that in the neigborhood of any point of V there exist local coodinates (p, q) in which the symplectic form is $dp \wedge dq$. Hence, in opposition to the case of Riemannian metrics, a sypmplectic form has no local invariant.

A fonction $H : V \times \mathbb{R} \to \mathbb{R}$ is called a it Hamiltonian ;

Definition 16 *The* symplectic gradient $\mathrm{grad}_\omega h$ *of $h : M \to \mathbb{R}$ is uniquely defined by the identity*

$$i_{X_h^*}\omega \;=\; -dh,$$

where ω is the symplectic form. It is also called the Hamiltonian vector field *defined by h. A time dependent Hamiltonian $H : M\times\mathbb{R}\to\mathbb{R}$, must be considered as a family, parametrized by t of functions $x\mapsto H(t,x)$, which leads to a time-dependent vector field X^*_H, uniquely defined by the identity*

$$i_{X^*_H}\omega = -\partial H := -dH + \frac{\partial H}{\partial t}dt.$$

The name symplectic gradient is of course given by analogy with the gradient of a Riemannian metric g on M, the only difference being that ω is antisymmetric while g is symmetric. If $\dim V = 2n$, one speaks of a *Hamiltonian system with n degrees of freedom* (resp. $n+\frac{1}{2}$ if H depends on time).

2.3.2 Examples : 1) Symplectic product and hermitian product : the standard symplectic form on $\mathbb{C}^n$. Identify $\mathbb{R}^{2n}$ to $\mathbb{C}^n$ by the mapping

$$(p,q) = (p_1,\cdots,p_n,q_1,\cdots,q_n)\mapsto \xi = (p_1+iq_1,\cdots,p_n+iq_n).$$

The euclidean scalar product $g((p',q'),(p'',q'')) = \sum_j(p'_jp''_j + q'_jq''_j)$ and the standard symplectic form $\omega((p',q'),(p'',q'')) = \sum_j(p'_jq''_j - q'_jp''_j)$ on $T^*\mathbb{R}^n\equiv\mathbb{R}^{2n}$ become respectively the real and imaginary parts of the canonical hermitian product:

$$(\xi_1,\xi_2,\ldots,\xi_n)\cdot(\eta_1,\eta_2,\ldots,\eta_n) = \sum_{j=1}^n\bar{\xi}_j\eta_j = g(\xi,\eta)+i\omega(\xi,\eta).$$

In particular, $\omega(\xi,\eta) = g(i\xi,\eta)$. Considered as an operator in $\mathbb{R}^{2n}$, the multiplication by i is represented in the coordinates (p,q) by a matrix J such that $J^2 = -Id$ and the above formulæ become

$$\forall X,Y\in\mathbb{R}^{2n},\ \omega(X,Y) = g(JX,Y),\ \text{where } J = \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}.$$

2) Vortices. As an illustration, let us write the classical equations *of n vortices* in the plane, that is the discretization due to Helmholtz of the *Euler equations* of the hydrodynamics (see the book by V.I. Arnold and B.A. Khesin, *Topological Methods in Hydrodynamics*, Springer 1998):

$$k_i\dot{x}_i = -\frac{\partial H}{\partial y_i},\quad k_i\dot{y}_i = \frac{\partial H}{\partial x_i},\quad H = \frac{1}{\pi}\sum_{i<j}k_ik_j\log r_{ij},$$

where $\mathbb{R}_{ij} = \sqrt{(x_i-x_j)^2+(y_i-y_j)^2}$ is the euclidean distance between the vortices i and j. The vector field is the symplectic gradient of H for the symplectic form $\varpi = \sum_i k_idx_i\wedge dy_i$. Here, the symplectic structure of $\mathbb{R}^2$ does not come from the identification with the cotangent bundle cotangent of $\mathbb{R}$, but from the identification to $\mathbb{C}$ (different for each vortex) of the plane $\mathbb{R}^2$ where the motion take place. This is a good example of a Hamiltonian system which does not come as the Legendre transform of a Lagrangian.

3.2 Three fundamental properties of a Hamiltonian flow on a symplectic manifold

The following properties of the flow $\varphi_{t_0}^{t_1}$ (resp. φ_t if H is autonomous) of a Hamiltonian vector field X_H^* are formal consequences of its expression as a symplectic gradient. The generalize to arbitrary symplectic manifolds what we already knew in the case of a cotangent bundle.

WARNING: *the last two properties hold only for autonomous Hamiltonians.*

1) Preservation of the symplectic form. For any times t_0, t_1 such that $\varphi_{t_0}^{t_1}$ is defined, one has

$$(\varphi_{t_0}^{t_1})^*\omega = \omega.$$

If $(V,\omega) = (T^*M, d\lambda)$, this was shown as a consequence of the existence of the integral invariant. In the general case, this is implied by the condition $d\omega = 0$. Indeed, using Cartan formula $L_X = i_X d + d i_X$ for the Lie derivative, one computes

$$L_{X_H^*}\omega = i_{X_H^*}d\omega + d i_{X_H^*}\omega = -\partial\partial H = 0.$$

2) Preservation of the energy in the autonomous case. Already proved twice for the Legendre tranforms of regular autonomous Lagrangians, it is intuitive on figure 13. Formally, it results from the identity

$$L_{X_H^*}H = dH(X_H^*) = \omega(X_H^*, X_H^*) = 0.$$

3) Characteristic foliation of a hypersurface. An important feature of autonomous Hamiltonian systems is that, *up to the parametrization, integral curves of the flow of X_H^* are completely determined by the sole geometry of the level hypersurfaces of H* : this is clear on figure 3.1 : the direction of $\mathrm{grad}_\omega H$ depends only on the direction of $\mathrm{grad}\, H$ and not on its length or orientation.

Definition 17 *Let (V,ω) be a symplectic manifold, Σ a regular hypersurface of V. The* characteristic foliation of Σ *is the dimension one foliation generated by the kernel of the restriction to Σ of the symplectic 2-form ω.*

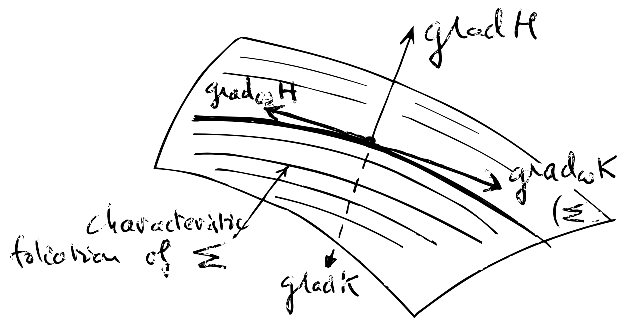


Figure 3.1 (H and K are regular equations of $H^{-1}(h) = K^{-1}(k)$ at x)

In other words, for any regular equation $H = 0$ of Σ in the neighborhood of the point z, the leaf through z coincides with the integral curve through z of the Hamiltonian vector field $\mathrm{grad}_{\varpi} H$.

3.3 Symplectic diffeomorphisms

An advantage of the Hamiltonian setting on the Lagrangian one is a greater flexibility of the changes of coordinates which do not spoil the canonical form of the equations.

Definition 18 *In a symplectic manifold (V, ω), one calls* symplectic *or* canonical *the diffeomorphisms (possibly local) $f : V \to V$ such that $f^*\omega = \omega$. More generally, a difféomorphism f from a symplectic manifold (V, ω) to another one (V', ω') is said symplectic if $f^*\omega' = \omega$.*

In $(T^*\mathbb{R}^n, dp \wedge dq)$, if $f(p_1, \cdots p_n, q_1, \cdots q_n) = (\alpha_1, \cdots \alpha_n, \beta_1, \cdots \beta_n)$, the condition becomes

$$\sum_{i=1}^{n} d\alpha_i \wedge d\beta_i = \sum_{i=1}^{n} dp_i \wedge dq_i.$$

The sum of the oriented areas of the projections of a domain on the 2-planes (p_i, q_i) is preserved; in particular, the symplectic diffeomorphisms of $T^*\mathbb{R}^n \equiv \mathbb{R}^{2n}$ preserve the 2n-form ω^n, hence also the volume $dp_1 \wedge \cdots \wedge dp_n \wedge dq_1 \cdots \wedge dq_n$ and the orientation (in mechanics, it is the classical *Liouville theorem*).

Exercices. 1) The symplectic group. Check that a linear isomorphism of $\mathbb{R}^{2n}$ is symplectic (for the standard symplectic form) if and only if the corresponding matrix A in the canonical basis satisfies

$${}^{tr}AJA = J, \text{ où } J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Check that the inverse and the transpose of a symplectic matrix are symplectic (for the transpose, there is implicitly the identification of $(\mathbb{R}^{2n})^*$ ̀to $\mathbb{R}^{2n}$ given by the canonical euclidean structure and the fact that the symplectic structure on $(\mathbb{R}^{2n})^*$ is given by the matrix $J^{-1} = -J$). The subgroup of $GL(2n, \mathbb{R})$ formed by the symplectic isomorphisms is called the *symplectic group*; it is denoted $Sp(2n)$. Notice the analogy with the *orthogonal group* $O(2n)$, where J (which defines the multiplication by i in $\mathbb{C}^n = \mathbb{R}^{2n}$) is replaced by the Identity I. Finally, deduce from what was said above on the Hermitian product that the following characterization of the *unitary group* $U(n)$:

$$U(n) = O(2n) \cap Sp(2n).$$

2) Check that φ is a symplectic diffeorphism of $\mathbb{R}^{2n}$ (for the standard symplectic form) if and only if, for all $x = (p, q)$, its derivative $d\varphi(x)$ belongs to $Sp(2n)$.

3) Cotangent maps. Let $\varphi : M \to N$ be a diffeomorpism and let $T\varphi$ and $T^*\varphi$ be its tangent and cotangent maps: if $\xi \in T_x M$ and $\alpha \in T_x^* M$,

$$T\varphi(\xi) = d\varphi(x)\xi, \quad T^*\varphi(\alpha) = \alpha \circ d\varphi(x)^{-1}.$$

Check that $T^*\varphi$ is a symplectic diffeomorphism if T^*M and T^*N are endowed xith their natural symplectic forms $\omega = d\lambda$ and $\omega' = d\lambda'$. Check that, in fact, $(T^*\varphi)^*\lambda' = \lambda$.

Applying this to the linear case, one gets the simplest class of elements of $Sp(2n)$, namely, the matrices of the form

$$A = \begin{bmatrix} B & 0 \\ 0 & {}^{tr}B^{-1} \end{bmatrix}, \quad \text{where} \quad B \in GL_n(\mathbb{R}).$$

Show that if $L : TM \to \mathbb{R}$ is a regular Lagrangian, the direct image by Tf of the Euler-Lagrange equations associated to L are the Euler-Lagrange equations associated to $L' = L \circ (Tf)^{-1}$: one shall show that if $\Lambda : TM \to T^*M$ and $\Lambda' : TN \to T^*N$ are the Legendre diffeomorphisms, the following diagram is commutative:

$$\begin{array}{ccc} TM & \xrightarrow{T\varphi} & TN \\ \downarrow{\scriptstyle\Lambda} & & \downarrow{\scriptstyle\Lambda'} \\ T^*M & \xrightarrow{T^*\varphi} & T^*M' \end{array}$$

4) Transformation of a Hamiltonian vector field by a symplectic diffeomorphism. Let $\Phi : (V_1, \omega_1) \to (V_2, \omega_2)$ be a symplectic diffeomorphism. Show that if $H_2 : V_2 \to \mathbb{R}$ is a Hamiltonian and $H_1 = H_2 \circ \Phi : V_1 \to \mathbb{R}$ is its inverse image by Φ, one has:

$$\text{grad}_{\omega_2} H_2 = \Phi_* \text{grad}_{\omega_1} H_1.$$

In particular, If $\Phi(p, q) = (a, b)$ is symplectic, that is if $dp \wedge dq = da \wedge db$ (or more correctly $\Phi^*(\omega) = \omega$), the direct image of the Hamiltonian vector field X_H^* is the Hamiltonian vector field $X^*_{H \circ \Phi^{-1}}$.

5) Generating functions. Here is a natural way of constructing symplectic diffeomorphisms: Let $S(a, q) : \mathbb{R}^n \times \mathbb{R}^n \cdots \to \mathbb{R}$ be such that $\frac{\partial^2 S}{\partial a \partial q}(a, q)$ is invertible on its domain of definition. Show that the formulæ

$$p = \frac{\partial S}{\partial q}(a, q), \quad b = \frac{\partial S}{\partial a}(a, q)$$

define a symplectic diffeomorphism

$$\Phi : \mathcal{O}_1 \ni (p, q) \mapsto (a, b) \in \mathcal{O}_2$$

from an open set of $\mathbb{R}^{2n}$ onto another one (both endowed with their canonical symplectic form).

6) Symplectic maps. Show that if $f : (V, \omega) \to (V', \omega')$ is symplectic, i.e. such that $f^*\omega' = \omega$is symplectic, it is necessarily an immersion (i.e. its derivative at each point is injective). *Indication :* show that if a tangent vector belongs to the kernel of $df(x)$, it must belong to the kernel of $\omega(x)$.

3.4 Lagrangian submanifolds and Hamilton-Jacobi

In the exemples of geodesic flows on the 2-torus which we studied in sections 1.4 and 1.5, most of the *phase space* $T^*\mathbb{T}^2$ is foliated by invariant tori on which the flow of X_H^* is a flow of translations in well chosen coordinates. This was obvious for the flat torus and a consequence of the invariance under rotation for the torus of revolution. The existence of such a foliation is a characteristic feature of the so-called *completely integrable* autonomous Hamiltonian systems. It follows from Lemma 10 that these invariant tori are very special: the symplectic 2-form ω vanishes identically on them. Such submanifolds play a fundamental role in higher dimension

Notation. *If $j : V \to M$ is the canonical inclusion of a submanifold and ω is a differential form on M, the pull-back $j^*\omega$ will be called the restriction of ω to V and denoted $\omega|_V$.*

Definition 19 *A submanifold V of a symplectic manifold (M, ω) is called isotropic if the restriction $\omega|_V$ of the symplectic form is identically zero.*

Lemma 20 *Let $(M, \omega = d\lambda)$ be an exact symplectic manifold. If the restriction of the flow of X_H^* to an invariant torus $\mathcal{T}$ is conjugate to a flow of translations with dense orbits, $\mathcal{T}$ is isotropic.*

Proof. If $j^*\omega = \sum_{i<j} a_{ij}(u_1, \cdots, u_k) du_i \wedge du_j$ in coordinates $u_1, \cdots, u_k$ on $\mathcal{T}$ such that the flow of X_H^* becomes a flow of translations $\Phi_t(u) = u + tv$, the fact that $\Phi^*\omega = \omega$ implies that the functions a_{ij} are constant along the integral curves contained in $\mathcal{T}$ *(this would not be the case if $d\Phi_t(u)$ was not the Identity).* As these integral curves are dense, the a_{ij} are constant, hence equal to 0 because $j^*\omega = d(j^*\lambda)$ is a coboundary.

Notice that in the completely integrable cases that we studied above, an easy argument of continuity implies that all invariant tori (and not only the ones with dense integral curves), and also the stable = unstable invariant manifolds of the hyperbolic periodic solutions, share the property $j^*\omega = 0$.

Definition 21 (Lagrangian submanifold) *Let (M, ω) be a symplectic manifold. The dimension of an isotropic submanifold of V is at most $\frac{1}{2}\dim M$. If it is equal to $\frac{1}{2}\dim M$, the submanifold is called Lagrangian.*

The bound on the dimension is an exercise in symplectic algebra : at each point $m \in M$, the bilinear form $\omega(m)$ is non degenerate, hence an isotropic subspace (i.e. a linear subspace contained in its ω-orthogonal) is at most of dimension $\frac{1}{2}\dim T_m M = \frac{1}{2}\dim M$.

If the symplectic manifold M is a cotangent bundle, that is if $M = T^*V$ endowed with the standard symplectic form, that is the derivative of the Liouville form: $\omega = d\lambda$, an example of a Lagrangian submanifold is the graph $\{(q, p = ds(q)), q \in V\}$ of the derivative of a differentiable function $s : V \to \mathbb{R}$. More generally, *Lagrangian submanifolds of $(T^*V, d\lambda)$ can be thought, at least locally, as graphs of derivatives of multiform differentiable functions.*

Remark. A Hamiltonian flow is a very particular one as it preserves the symplectic 2-form ω, hence in particular the volume. *Its restriction to a Lagrangian submanifold V, on the contrary, does not satisfy any a priori constraint : every vector field X on V is the restriction of a Hamiltonian flow defined on a neighborhood of V.* The simplest example is obtained when $V \equiv \mathbb{T}^n$ is the zero-section $p = 0$ of $T^*\mathbb{T}^n$: if $X(q)$ is vector field on V, the Hamiltonian $H(p,q) = p \cdot X(q)$ is such that the restriction of X_H^* to V coincides with X (but it is not convex in p !).

Each invariant Lagrangian submanifold that we found in the integrable examples is contained in a single energy level. This is a consequence of the conservation of energy when the submanifold is the closure of a single solution and the others follow by continuity. This property has a very important converse :

Proposition 22 *Let $H : M \to \mathbb{R}$ be an autonomous Hamiltonian on the symplectic manifold (M, ω). Every Lagrangian submanifold V of M contained in a regular energy level $H^{-1}(h)$ is invariant under the flow of X_H^*.*

The proof is again an exercise in symplectic algebra : because of the maximality of the dimension of V among isotropic submanifolds, it is enough to notice that at each point $m \in H^{-1}(h)$, the vector $X_H^*(m)$ belongs to (in fact generates) the kernel of $i_h^*\omega(m)$, where i_h is the canonical injection of $H^{-1}(h)$ in M. Indeed, if $X_H^*(m)$ was not contained in T_mV, the linear subspace generated by $X_H^*(m)$ and T_mV would be isotropic of dimension $n+1$, a contradiction.

In case $V = T^*M$ is a cotangent, it is a so-called Hamilton-Jacobi equation which expresses that the graph ds of the derivative of a differentiable function $s : M \to \mathbb{R}$ is contained in some energy level :

Definition 23 (Hamilton-Jacobi) *The time-independent Hamilton-Jacobi equations associated to the Hamiltonian $H(p,q)$ are the equations of the form*

$$H(ds(q), q) = h.$$

The time-dependent Hamilton-Jacobi equation associated to the Hamiltonian $H(p,q,t)$ is the equation

$$\frac{\partial S}{\partial t}(q,t) + H\big(\frac{\partial S}{\partial q}(q,t), q, t\big) = 0.$$

Of course, after identification of $K^{-1}(0)$ with $T^*M \times \mathbb{R}$, the time dependent equation is nothing but the time-independent one, $K(dS(q,t), q, t) = 0$, where K is defined by $K(p, E, q, \tau) = E + H(p, q, \tau)$.

Proposition 22 is at the basis of the *method of characteristics*: given an autonomous Hamiltonian H on a $2n$-dimensional symplectic manifold V, if $\mathcal{I} \subset H^{-1}(h)$ is an $n-1$ dimensional isotropic submanifold which is transverse to the integral curves of X_H^*, the union of those integral curves which meet $\mathcal{I}$ is a Lagrangian submanifold contained in $H^{-1}(h)$. In case V is a cotangent $V = T^*M$, $\mathcal{I}$ can be chosen as a Cauchy datum, that is a graph over a hypersurface $\mathcal{F} \subset M$

of the derivative ds of a differentiable function $s : \mathcal{F} \to \mathbb{R}$. However, in general only local (in time) solutions do exist. In order to obtain globally defined solutions, one has to abandon differentiability and accept Lipschitz solutions. This is the theory of shocks in scalar conservation laws or more generally the *Weak KAM theory*. However, there are cases where global differentiable solutions do exist, namely the"completely integrable" systems and their perturbations (*K.A.M. theory*).

3.5 Complete integrability in $T^*\mathbb{T}^n$

All invariant tori of the geodesic flow of a flat torus are graphs of a mapping $q \mapsto p(q)$, that is sections of the projection $(p, q,) \mapsto q$. For the torus of revolution, only those not contained in the resonance zone are graphs in the same way. The invariant manifolds of the hyperbolic periodic solutions are the union of two pieces, each of which is a graph.

Lemma 24 *If the Lagrangian submanifold $\mathcal{L}$ of $V = T^*\mathbb{T}^n = (\mathbb{R}^n)^* \times \mathbb{T}^n$ is a graph, it is the graph of a mapping of the form $p = a + ds(q)$, where $a = (a_1, \cdots, a_n) \in (\mathbb{R}^n)^*$ and $s : \mathbb{T}^n \to \mathbb{R}$.*

The proof is an easy calculation : the graph V of the mapping $q \mapsto p(q)$ is Lagrangian if and only if the 2-form $\sum_{i=1}^n dp(q) \wedge dq = \sum_{i,j} \frac{\partial p_i}{\partial q_j}(q) dq_j \wedge dq_i$ on $\mathbb{T}^n$ is identically 0. But this means that $\frac{\partial p_i}{\partial q_j}(q) = \frac{\partial p_j}{\partial q_i}(q)$ for all i, j. This implies that there exists a function $\sigma : \mathbb{R}^n \to \mathbb{R}$ such that for all i, $p_i(q) = \frac{\partial \sigma}{\partial q_i}(q)$. Hence there exist constants a_i (the *periods* of σ) and a function $s : \mathbb{T}^n \to \mathbb{R}$ such that for all i, $p_i(q) = a_i + \frac{\partial s}{\partial q_i}(q)$.

Corollary 25 *A Lagrangian graph $\mathcal{L}$ contained in the energy level $H^{-1}(h)$ of an autonomous Hamiltonian is of the form $\{(p, q), p = a + ds\}$, where s is a solution of the partial differential equation $H(a + ds(q), q) = h$.*

According to the above Corollary, each Lagrangian graph contained in an energy level of an autonomous Hamiltonian is the graph of the derivative of a solution of the Hamilton-Jacobi equation associated to a Hamiltonian

$$H_a(p, q) = H(a + p, q)$$

where $a \in (\mathbb{R}^n)^*$ should actually be thought of as a cohomology class in $\mathcal{H}^1(\mathbb{T}^n, \mathbb{R})$. Such a Hamiltonian is easily seen to be the Legendre transform of the Lagrangian

$$L_a(q, \dot{q}) = L(q, \dot{q}) - a \cdot \dot{q} = L(q, \dot{q}) - \sum_{i=1}^n a_i \dot{q}_i,$$

which satisfies the same hypotheses as the original one.

Notice that, *while the solutions of the Euler-Lagrange equations associated to L_a are independent of a, the minimizing ones do indeed depend on a.* The simplest example is the geodesic flow of the flat torus : adding a mass to better distinguish between the tangent and cotangent sides, let us take $L(q,\dot q) = \frac{m}{2}||\dot q||^2$. The Lagrangian L_a can be written

$$L_a(q,\dot q) = \frac{m}{2}||\dot q||^2 - a\cdot\dot q = \frac{m}{2}||\dot q - \frac{1}{m}a||^2 - \frac{||a||^2}{2m},$$

and the minimizers are immediately seen to be such that $\dot q = \frac{1}{m}a$, that is $p = a$.

Let us suppose now that to each $a \in (\mathbb{R}^n)^*$ we can associate in a differentiable way a solution u_a of the equation

$$H(a + du_a(q), q) = h(a),$$

where h is a smooth function. Setting $S_0(a,q) = a\cdot q + u_a(q)$, this is equivalent to

$$H\left(\frac{\partial S_0}{\partial q}(a,q), q\right) = h(a).$$

The function S_0 is of course not $\mathbb{Z}^n$-periodic in q, that is not defined on $\mathbb{T}^n$, but its derivative is. Hence S_0 can be used as the *generating function* of the *symplectic transformation*

$$\Phi : \mathbb{R}^n \times \mathbb{T}^n \to \mathbb{R}^n \times \mathbb{T}^n,\ \ \Phi(p,q) = (a,b),$$

defined by

$$p = \frac{\partial S_0}{\partial q}(a,q) = a + \frac{\partial u_a}{\partial q}(q), \quad b = \frac{\partial S_0}{\partial a}(a,q) = q + \frac{\partial u_a}{\partial a}(q),$$

provided $\det\left(\frac{\partial^2 S_0}{\partial a \partial q}\right) \neq 0$. Indeed, $dp\wedge dq + db\wedge da = d^2S_0 = 0$, which shows the preservation of the canonical symplectic form. In the new coordinates (a,b), the Hamiltonian vector field X_H^* becomes $X^*_{H\circ\Phi^{-1}}$, that is X_h^*. As h does not depend on the variables b, Hamilton's equations take the particularly simple *completely integrable* form

$$\frac{da_i}{dt} = 0,\ \ \frac{db_i}{dt} = \frac{dh}{da_i}(a),$$

which is similar to the one defining the geodesic flow of the flat torus.

This is not astonishing. If for each a there exists a unique solution u_a which is differentiable, the collection of the graphs of these functions defines a foliation of the phase space by Lagrangian tori. The existence of such a foliation implies in turn the existence of *action-angle* coordinates in which the flows on the invariant tori are linear.

4 Two degrees of freedom: surfaces of section and return maps

4.1 Poincaré return maps

Due to its invariance under rotation, the geodesic flow of the torus of revolution (or the one of the flat torus) admits a quotient (i. e. a direct image) on the torus (ψ, θ); of course, such a quotient does not exist any more for the geodesic flow of a slightly perturbed metric which is no more invariant under rotation. What replaces the non-existing quotient is a *surface of section*[3] analogous to the one discovered by Poincaré in the *Restricted problem of three bodies* which will be described in section 6. More precisely, the torus (ψ, θ) may be identified to the submanifold $\mathcal{S} \subset T^1\mathbb{T}^2$ whose equation is $\varphi = 0$; indeed, when restricted to this submanifold, the quotient map $(\varphi, \psi, \theta) \mapsto (\psi, \theta)$ is a diffeomorphism. If one excludes the family of periodic solutions $\varphi = \varphi_0,\ \theta = \frac{\pi}{2}$ modulo π, each integral curve of the geodesic flow meets $\mathcal{S}$ transversally. Let γ_1 and γ_2 be the two members of the above family which belong to $\mathcal{S}$. On the surface $\mathcal{S}\setminus\{\gamma_1 \cup \gamma_2\}$, which is diffeomorphic to the disjoint union of two open annuli, one can define a *first return map* (or *Poincaré map*) P which sends the point x on the first point of the integral curve (described from x in the positive direction) which belongs again to $\mathcal{S}$. The mapping P is a diffeomorphism of $\mathcal{S} \setminus \{\gamma_1 \cup \gamma_2\}$ onto itself; iterating it can be viewed as a *stroboscopy* of the dynamics: its knowledge determines the topology (but not the parametrization) of the phase portrait in $T^1\mathbb{T}^2$ of the geodesic flow. This leads us to the

Definition 26 . *A local* surface of section *at x of a vector field X is a submanifold $\mathcal{S}$ which contains x and has the following properties:*
1) It is everywhere transverse to the integral curve of X through this point;
2) There exists an open neighborhood $\mathcal{O}$ of x in $\mathcal{S}$ on which the first return map P is defined (figure 4.1).
A global surface of section *is a submanifold possibly containing a finite number of integral curves of X and which, in the complement of these curves, is everywhere a local surface of section*

The following lemma asserts that the discrete analogue of a Hamiltonian vector field X is a symplectic diffeomorphism:

Lemma 27 . *Let (N, ω) be a symplectic manifold, H a regular function on N, $X_H^* = \text{grad}_\omega H$ its symplectic gradient, $X_{H,h}^*$ the restriction of X_H^* to the regular energy level Σ_h. Let $\mathcal{S} \subset \Sigma_h$ be a local surface of section of $X_{H,h}^*$, $P : \mathcal{O} \to \mathcal{S}$ the corresponding first return map. The 2-form $\omega_{\mathcal{S}}$ induced on $\mathcal{S}$ by the symplectic form ω is non-degenerate (and hence symplectic) and invariant under P.*

Proof. The non-degeneracy of $\omega_{\mathcal{S}}$ comes from the fact that $X_{H,h}^*$ generates the kernel of the form ω_h induced by ω on Σ_h. Let φ_t denote the flow of $X_{H,h}^*$; the

[3] Among the basic operations which can be used by the geometer, two main ones are project and cut.

first return map can be written $P(x) = \varphi_{t(x)}(x)$, where $x \in \mathcal{O}$, and $t(x) > 0$ are the *first return time* on $\mathcal{S}$. As the transversality hypothesis insures the regularity of $t(x)$, one deduces that, if A is tangent at x to $\mathcal{S}$,

$$d\varphi_{t(x)}(x)A = dP(x)A + \alpha X_H^*\big(P(x)\big).$$

The conclusion follows from the preservation of ω by the flow φ_t and the fact that $X_{H,h}^*$ belongs to the kernel of ω_h.

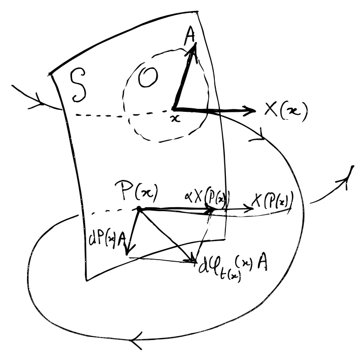


Figure 4.1

The invariant tori become invariant curves and complete integrability corresponds to the the existence of a (possibly singular) foliation of $\mathcal{S}$ by invariant curves.

In the following, we give simple examples and introduce the *monotone twist maps* which will be studied in section 5, postponing to section 6 the study of the *restricted problem of three bodies* which is at the origin of the introduction by Poincaré of the notion of surface of section.

Remark. Given two hypersurfaces P_0 et P_1 in Σ_h (i.e. $2n-2$ dimensional sub manifolds of V) which are transversal to the integral curves of the restriction $X_{H,h}^*$ of X_H^* to Σ_h in some domain of Σ_h, one defines the *Poincaré map* $\mathcal{P} : P_0 \to P_1$ in the following way: the image of $x \in P_0$ is first point of P_1 which one encounters when following the integral curve starting at x.

Exercise. Deduce from the characterization of the characteristic foliation that ω induces on P_0 and P_1 symplectic forms ω_0 and ω_1 and that the map $\mathcal{P}$ is symplectic, that is $\mathcal{P}^*\omega_1 = \omega_0$.

4.2 Time periodic Hamiltonians

A particularly simple case is provided by the autonomous Hamiltonians

$$K(p,E,q,t) = E + H(p,q,t) : T^*(M \times \mathbb{T}^1) \to \mathbb{R}$$

originating from a non-autonomous Hamiltonian $H(p,q,t) : T^*M \times \mathbb{R} \times \mathbb{T}^1 \to \mathbb{R}$ periodic in t. In the energy hypersurface $K = 0$, diffeomorphic to $T^*M \times \mathbb{T}^1$ (see section 2.4), fixing $t = 0$ provides a natural hypersurface of section diffeomorphic to T^*M.

4.3 Geodesic flows on the torus and on the sphere

The torus. In the case of a flat torus or a torus of revolution, the restriction of the symplectic form to the torus (ψ,θ) is, up to a factor r in the second case, the measure $\cos\theta\, d\theta\wedge d\psi$ (exercise, check this !). The integral curves of the flow after a quotient by $SO(2)$ (i.e. in the torus (ψ,θ)) are defined by the equations $C(\psi,\theta)=$ cste, where $C(\psi,\theta)=\theta$ if the torus is flat, $C(\psi,\theta)=(1+r\cos\psi)\cos\theta$ (the Clairaut integral) if the torus is a torus of revolution. It follows that the mapping P, whose orbits are contained in the integral curves of the quotient flow, satisfies

$$P(\psi,\theta)=P\big(C(\psi,\theta)\big).$$

One says that the level curves of C are *invariant curves* of the mapping P. Their existence corresponds to the *complete integrability* of the above geodesic flows. Figure 4.2 shows these invariant curves in the annulus $-\pi/2<\theta<\pi/2$ for the flat torus and the torus of revolution. One has represented in both cases the image by P of the ray $\mathcal{R}$ with equation $\psi=0$: a "monotone twisted" ray in the first case, more complex (non monotone) in the second one because the geodesics close to the exterior "horizontal" periodic geodesic are focalized beween two successive returns on the annulus of section[4] (analogous to the iterates of a monotone distortion). This torsion of the rays is related to the non-degeneracy of the Hamiltonian and hence to the *Legendre property* $\frac{\partial^2 L}{\partial \dot q^2}>0$ (see section 5.3). It corresponds to the effective variation of the "frequencies" with the "actions", when the energy is fixed. Such mappings were the object of numerous studies since Poincaré and Birkhoff (see section 5);

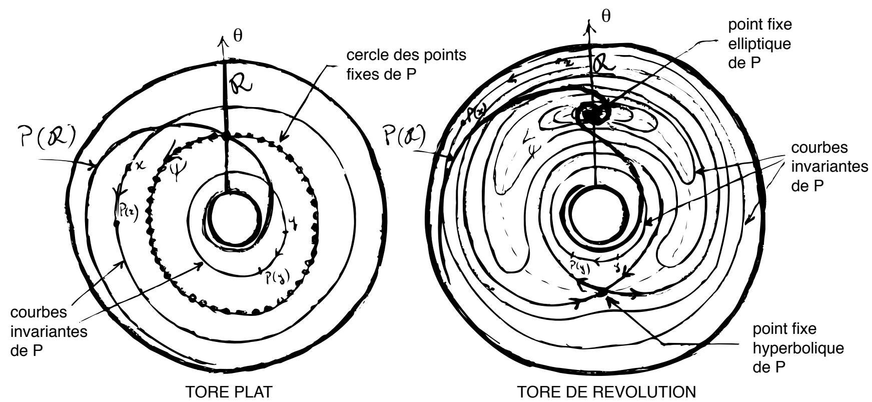


Figure 4.2

When going[5] from the flat torus to the torus of revolution, the circle of fixed points $\theta=0$ of the first return map P has blowned up into a pair *elliptic-hyperbolic* of isolated fixed points: these two types of fixed points can be distinguished by the derivative of P, conjugated to a rotation in the elliptic case,

[4]Looking again to the end of 1.5, the reader will notice that this affirmation follows from the inequality $\pi\sqrt{r(1+r)}<2\pi(1+r)$.

[5]recall, what is obvious on figure 4.2, that it is not a small perturbation.

with two real eigenvalues λ_1, λ_2 such that $0 < |\lambda_1| < 1 < |\lambda_2|$ in the hyperbolic case (the conservation of orientation and mesure $\cos\theta\, d\psi \wedge d\theta$ implies moreover the equality $\lambda_1\lambda_2 = 1$). Notice that the hyperbolic fixed point appears at the local level as the intersection of two invariant curves, the *stable manifold* and the *unstable manifold*, non-linear avatars of the eigenspaces of the derivative at P, respectively associated to the eigenvalues λ_1 et λ_2.

WARNING. Let X be a C^∞ (or analytical) vector field on the 2-torus. Suppose X admits a circle of section $\mathcal{S}$ and that the first return map P on this circle is conjugate to a rotation. Then, it exists global angular coordinates on the torus such that the integral curves of X are parallel straight lines (as in the right part of figure 1.9). But this does not mean that there exists global angular coordinates which make X a contant vector field. Indeed, the existence of such coordinates depends on the *diophantine properties* of the rotation number of P, that is on the way this angle is aproximated by rational multiples of 2π (see for example the first chapter of the second volume of S. Sternberg's book *Celestial Mechanics*.

The sphere. The unit tangent bundle of the unit 2-sphere $S^2 \subset \mathbb{R}^3$ can be identified to the rotation group $SO(3)$ by sending the unit tangent vector Y (identified by translation to a vector in $\mathbb{R}^3$) at the point $X \in S^2$ on the unique rotation $R \in SO(3)$ which sends the canonical basis of $\mathbb{R}^3$ on the triple $(X, Y, X \wedge Y)$ (see the figure). *It will be convenient to denote by (X, Y) the tangent vector at the point $X \in S^2$ represented by Y.*

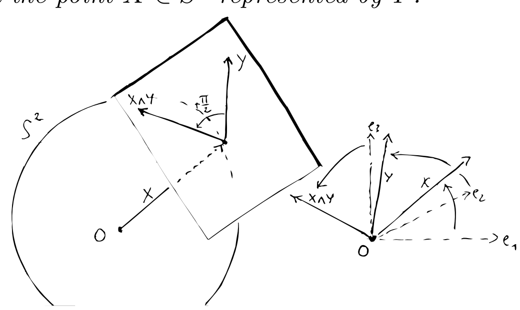


Figure 4.3

If we first endow the sphere with the round metric, induced on the unit sphere by the standard euclidean metric of $\mathbb{R}^3$, the geodesics are the great circles, followed at constant (=1) velocity (exercise). Translating tangent vectors at the origin, one identifies the orbit under the geodesic flow of the tangent vector $(X, Y) \in T^1S^2$ – that is the set of tangent vectors along the intersection of the sphere with the plane throuh the origin orthogonal to $X \wedge Y$ – with the unit tangent space at the point $X \wedge Y$.

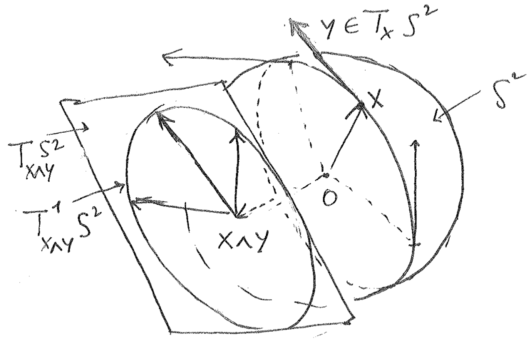


Figure 4.4

It follows that the map sending a tangent vector to its orbit under the geodesic flow is isomorphic to the the projection of T^1S^2 onto its base S^2 Its Euler characteristic being equal to 2, the sphere does not admit a nowhere vanishing vector field; in particular, its tangent bundle cannot be trivial. Hence it is not possible to find a surface of section of the geodesic flow of the round sphere (or of an almost round one) which cuts transversally every trajectory in one point. The Birkhof annulus of section for the restricted three-body problem described in section 6, is obtained by blowing up two points of S^2 into their orbit (see figure 6-2)

4.4 The Poincaré normal form around an elliptic fixed point and twist maps

Let $F : (S,p) \to (S,p)$ be a local C^∞ (or analytic) diffeomorphism of a surface S defined in the neighborhood of a fixed point $p = F(p)$. The fixed point is said to be elliptic if the spectrum of the derivative $dF(p)$ is of the form $\{2\pi i\omega, -2\pi i\omega\}$ with $\omega \neq \pm 1$. This is equivalent to the existence of a linear conjugation of $dF(p)$ with the rotation of angle $2\pi\omega$. Hence, after choosing good coordinates, one can suppose that $p = 0$ and that $F : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ is such that

$$F(\zeta) = \lambda\zeta + O(|\zeta|^2), \quad \text{with} \quad \lambda = e^{2\pi i\omega}.$$

In other words, F is a perturbation of a rotation.[6] Now, a rotation preserves each circle centered at the origin. This is a very strong property, very likely to be destroyed by the non-linear terms in the Taylor expansion of F. Nevertheless, reality is subtler and the study of the fate of these invariant circles is the starting point of two famous theories which correspond roughly to the dichotomy between *dissipative* and *conservative* dynamics (see [C3]):

1) *Andronov-Hopf-Neimark-Sacker bifurcation theory* which analyzes what hap-

[6]Beware that the notation $F(\zeta)$ does not mean that F is complex analytic, its expression depends on ζ and $\overline{\zeta}$

pens when one considers a *generic*[7] diffeomorphism F with an elliptic fixed point at 0. The local behaviour of F itself is quite dull: indeed, the radial behaviour of the nonlinear terms turns the fixed point into an attractor or a repulsor and no other invariant object persists in its neighborhood. It is only when considering "generic" 1-parameter families F_μ of local diffeomorphisms stemming from $F_0 = F$ that the whole richness of the dynamics is regained: each small enough circle invariant under the rotation $dF(0)$ becomes a *normally hyperbolic*[8] closed curve invariant under some F_μ.

2) *Kolmogorov-Arnold-Moser (K.A.M.) theory* which analyzes the case when F is area preserving, a hypothesis which, as we have seen, is natural for diffeomorphisms with a mechanical origin or more generally a Hamiltonian origin. In this case, it is the angular behaviour of the non-linear terms which plays the key part, the result being that "many" of the circles invariant under the rotation $dF(0)$ persist in the form of closed curves invariant under the action of F itself. Moreover the restriction of F to such an invariant closed curve is smoothly conjugated to a rotation whose angle is of the form $2\pi\alpha$ with α not rational and even "far from the rationals" in a precise sense.

The first insight, which goes back to Poincaré's thesis in 1879, is the following: being a rotation, the derivative of F commutes with the whole group $SO(2)$ of rotations. This is shown to imply that, provided some conditions on ω are satisfied, a high order approximation of F is locally invariant by an action of $SO(2)$ close to the standard one. Equivalently, one proves the existence of local coordinates which reveal the approximate geometry of the map, in a spirit similar to the Jordan form of a matrix:

Theorem 28 *If $\lambda = e^{2\pi i\omega}$ is such that $\lambda^q \neq 1$ for all integers $q \in \mathbb{N}$ such that $q \leq 2n+2$, there exists a local diffeomorphism*

$$H : (\mathbb{C}, 0) \to (\mathbb{C}, 0), \quad \zeta \mapsto z = H(\zeta) = \zeta + O(|\zeta|^2)$$

such that

$$H\circ F\circ H^{-1}(z) = N(z) + O(|z|^{2n+2}), \quad \text{where} \quad N(z) = z\left(1 + f(|z|^2)\right)e^{2\pi i(\omega+g(|z|^2))},$$

with f and g real polynomials of degree n such that $f(0) = g(0) = 0$. If moreover $\lambda^{2n+3} \neq 1$, one can achieve a rest which is $O(|z|^{2n+3})$.

The so-called *normal form* N, is characterized by the fact that it commutes with the whole group $SO(2)$ of rotations:

$$\forall\alpha, N(e^{2\pi i\alpha}z) = e^{2\pi i\alpha}N(z).$$

[7] We shall not give a formal definition of this word; it means essentially that what is described is the general situation and that only special hypotheses could prevent the description to be correct.

[8] Roughly speaking this mean that any attraction or repulsion normal to the curve under the iterates of F_μ dominates any attraction or repulsion inside the curve; this condition insures the robustness of the curve

Proof. Let us start with a local diffeomorphism of degree 2,

$$H_2 : (\mathbb{C}, 0) \to (\mathbb{C}, 0), \quad z = H_2(\zeta) = \zeta + \sum_{i+j=2} \gamma_{ij} \zeta^i \overline{\zeta}^j.$$

The direct computation of $H_2 \circ F \circ H_2^{-1}$ is illustrated on the diagram below:

$$\begin{tikzcd}
\zeta \arrow[r, "F"] \arrow[d, "H_2"] & \lambda\zeta + \sum_{i+j=2} \alpha_{ij} \zeta^i \overline{\zeta}^j + O(|\zeta|^3) \arrow[d, "H_2"] \\
\zeta + \sum_{i+j=2} \gamma_{ij} \zeta^i \overline{\zeta}^j \arrow[r, "H \circ F \circ H^{-1}"] \arrow[d, phantom, "\parallel"] & \lambda\zeta + \sum_{i+j=2} \alpha_{ij} \zeta^i \overline{\zeta}^j + \sum_{i+j=2} \gamma_{ij} \lambda^i \overline{\lambda}^j \zeta^i \overline{\zeta}^j + O(|\zeta|^3) \arrow[d, phantom, "\parallel"] \\
z & \lambda z + \sum_{i+j=2} \left(\alpha_{ij} + \gamma_{ij} (\lambda^i \overline{\lambda}^j - \lambda)\right) z^i \overline{z}^j + O(|z|^3)
\end{tikzcd}$$

Supposing that $F(\zeta) = \lambda\zeta + \sum_{i+j=2} \alpha_{ij} \zeta^i \overline{\zeta}^j + O(|\zeta|^3)$, we get

$$H_2 \circ F \circ H_2^{-1}(z) = \lambda z + \sum_{i+j=2} \Big(\alpha_{ij} + (\lambda^i \overline{\lambda}^j - \lambda)\gamma_{ij}\Big) z^i \overline{z}^j + O(|z|^3).$$

Hence, if no *resonance* relation of the form $\lambda^i \overline{\lambda}^j - \lambda = 0$ is satisfied with indices i, j such that $i + j = 2$, that is if $\lambda^3 \neq 1$ (otherwise $\overline{\lambda}^2 - \lambda = 0$), the choice of $\gamma_{ij} = -(\lambda^i \overline{\lambda}^j - \lambda)^{-1} \alpha_{ij}$ kills all degree 2 terms in the Taylor expansion of the transformed map $H_2 \circ F \circ H_2^{-1}$.
If one tries in the same way to simplify the terms of degree 3 in the Taylor expansion of $H_2 \circ F \circ H_2^{-1}$, one stumbles upon an *unavoidable resonance*

$$\lambda^2 \overline{\lambda} - \lambda = 0$$

which merely reflects that $|\lambda| = 1$. Hence , if no other resonance of order 3 exists, which amounts to saying that $\lambda^4 \neq 1$ (otherwise $\overline{\lambda}^3 - \lambda = 0$), a local diffeomorphism H_3 of the form $H_3(z) = z + \sum_{i+j=3} \gamma_{ij} z^i \overline{z}^j$ can be found such that[9]

$$H_3 \circ H_2 \circ F \circ H_2^{-1} \circ H_3^{-1}(z) = \lambda z + c_1 z|z|^2 + O(|z|^4).$$

Now, if $\lambda^q \neq 1$ for all $q \leq 2n + 3$, one finds by induction a local diffeomorphism $H = H_{2n+2} \circ H_{2n+1} \circ H_3 \circ H_2$ tangent to Id at 0 such that

$$H \circ F \circ H^{-1}(z) = \lambda z + \sum_{k=1}^{n} c_k z|z|^{2k} + O(|z|^{2n+3}).$$

[9] in order to avoid too cumbersome notations we still call z the transformed coordinate $H_3(z)$.

If $\lambda^{2n+3} = 1$, there is possibly a monomial $\gamma\bar{z}^{2n+2}$ which cannot be canceled. Finally, chosing polar coordinates, one writes $H \circ F \circ H^{-1}$ as in the conclusion of the theorem.

Remark. Resonances of the form $\lambda^q = 1$ for $1 \leq q \leq 4$ are called *strong resonances*. They are characterized by the fact that the resonant monomial $\bar{z}^{q-1}$ is of smaller or comparable order to the first unvoidable resonant monomial $z|z|^2$ and hence could play a role in the geometry of the normal form N which could become invariant only by rotations by an angle multiple of $2\pi/q$. *In the sequel, the hypotheses always exclude strong resonances.*

Theorem 28 allows us to suppose from the start that local coordinates z have been chosen so that F is in the form given, by Theorem 1. In other words, from now on we shall write $F(z)$ instead of $H \circ F \circ H^{-1}(z)$.
We now suppose that, in addition to satisfying $\lambda^q \neq 1$ for all integers $1 \leq q \leq 4$, F is area preserving. It follows that the radial component f of the normal form N vanishes identically and one can show that it is possible to choose H area preserving. Hence, one is reduced to the study in the neighborhood of its elliptic fixed point 0 of an area preserving diffeomorphism of $\mathbb{C}, 0$ of the form

$$F(z) = N(z) + O(|z|^4), \quad N(z) = ze^{2\pi i(\omega + b_1|z|^2)}.$$

The normal form N is called a *truncated Birkhoff normal form*. Dynamically, it is an *integrable monotone twist mapping*: as well as the rotation $dF(0)$, it leaves invariant each circle C_r centered at 0 but the angle of rotation $2\pi(\omega + b_1 r^2)$ on C_r varies now monotonically with the radius r of this circle

Poincaré, while studying the three body problem, became aware of a fundamental difference between the invariant circles on which N induces a periodic ($\omega + b_1 r^2$ rational) or non periodic ($\omega + b_1 r^2$ irrational) rotation: in the first case (angle $2\pi\omega = 2\pi p/q$) the invariant circle is simply the union of a continous family of q-periodic points z (i.e. of points z such that $N^q(z) = z$); in consequence, a small perturbation should in general break such a circle, with only a finite number of periodic points surviving the perturbation. On the other hand, if ω is irrational, the invariant circle being the closure $\overline{\cup_{n\geq 0} N^n(z)}$ of an orbit has a dynamical origin and hence has more chance to resist a perturbation. In the first volume of his famous book *The New Methods of Celestial Mechanics*, Poincaré even ventured to write that some arithmetic condition on ω could perhaps grant resistance to perturbations of such an invariant circle but that he considered such a possibility as quite improbable.

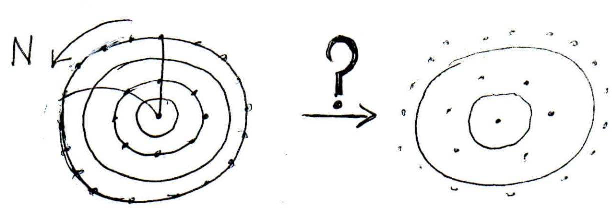


Figure 4.5. Perturbation of a monotone twist

Nevertheless, after the pioneering work of Kolmogorov in 1954, the so-called *K.A.M. theory* (from the names of Kolmogorov, Arnold and Moser) showed that indeed, what Poincaré deemed improbable was in fact a dominant phenomenon. In the present case, the pertinent statement is the following

Theorem 29 (Moser 1962) *Given an area preserving diffeomorphism F as above, given $C > 0$ and $\beta > 0$, there exists $\epsilon(C, \beta) > 0$ such that each invariant circle C_{r_0} of the normal form N such that its rotation angle $2\pi\omega_{r_0} = 2\pi(\omega + b_1 r_0^2)$ satisfies the diophantine condition*

$$\forall\, \frac{p}{q} \in \mathbb{Q},\ \left|\omega_{r_0} - \frac{p}{q}\right| \geq \frac{C|\omega_{r_0} - \omega|}{|q|^{2+\beta}} \quad \textit{and} \quad |\omega_{r_0} - \omega| < \epsilon(C, \beta)$$

will give rise to a smooth (resp. analytic) closed curve Γ_{r_0} invariant under F and such that the restriction $F|_{\Gamma_{r_0}}$ of F is smoothly conjugate to the rotation of angle $2\pi\omega_{r_0}$.

This theorem will be studied in the course by Bassam Fayad. In the next section, we shall address the problem of periodic points.

5 Monotone twists, periodic orbits and Mather sets

In this last part, we prove the existence of Birkhoff orbits and as a consequence the existence of Aubry-Mather invariant sets for any monotone area preserving twist map of the annulus. This generalizes Birkhoff results on the billard map (see [C1] section 1.4, figure 3). An important generalization to higher dimensions exists : this is the so-called Weak KAM theory.

5.1 Ordered invariant sets and Lipschitz estimates

Let $\overline{A}$ be the closed annulus $\mathbb{T}^1 \times [0, 1]$ (resp. the open cylinder $\mathbb{T}^1 \times \mathbb{R}$). Let $\overline{F} = (\overline{F}_1, \overline{F}_2) : \overline{A} \to \overline{A}$ be an orientation preserving C^k-diffeomorphism with $k \geq 1$. We call $(x \in \mathbb{T}^1,\, y \in \mathbb{R})$ the natural coordinates in $\overline{A}$. We shall note $F = (F_1, F_2)$ a lift of $\overline{F}$ to the universal covering $A = \mathbb{R} \times [0, 1]$ (resp. $\mathbb{R}^2$) of $\overline{A}$.

Definition 30 *The map F is said to be a positive monotone twist map if there exists a constant $a > 0$ such that both $\frac{\partial F_1}{\partial y}$ and and $-\frac{\partial (F^{-1})_1}{\partial y}$ are bounded below by a. If moreover F preserves the standard Lebesgue measure $dxdy$ (or more generally a measure which weights positively any open set) one says it is conservative.*

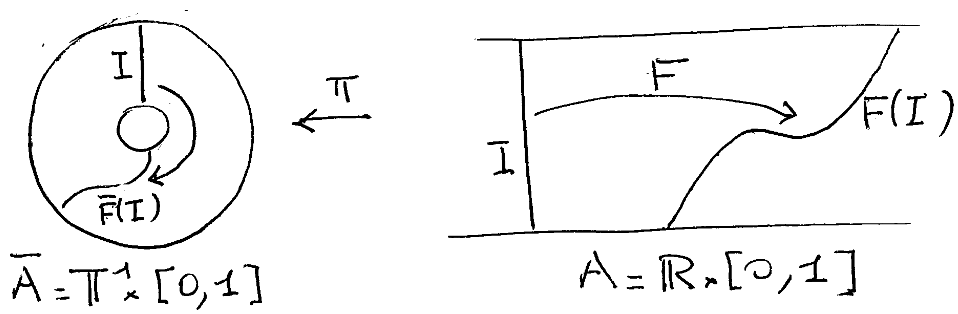


Figure 5.1

A simple example is

$$F : A \to A, \quad F(x,y) = (x+y, y).$$

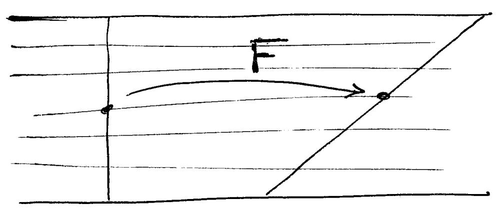


Figure 5.2

Another simple example is

$$F(x,y) = R_{p/q} \circ \varphi_t = \varphi_t \circ R_{p/q},$$

where $R_{p/q}(x,y) = (x + p/q, y)$ and φ_t is the flow at a small positive time t of the pendulum-type differential equation

$$\ddot{x} + \omega^2 \sin(2\pi q x).$$

The figure below, which illustrates the case $p = 2, q = 3$, features the level curves of the conserved "energy" $H(x,y) = \frac{1}{2}y^2 - \frac{\omega^2}{2\pi q}\cos(2\pi q x)$. To the singular points of H correspond two isolated untertwined periodic orbits of period 3 and rotation number $p/q = 2/3$, one *hyperbolic*, $\{z_0, F(z_0), F^2(z_0)\}$, corresponding to an unstable equilibrium of the pendulum, and one *elliptic*, $\{z_1, F(z_1), F^2(z_1)\}$, corresponding to a stable equilibrium of the pendulum.

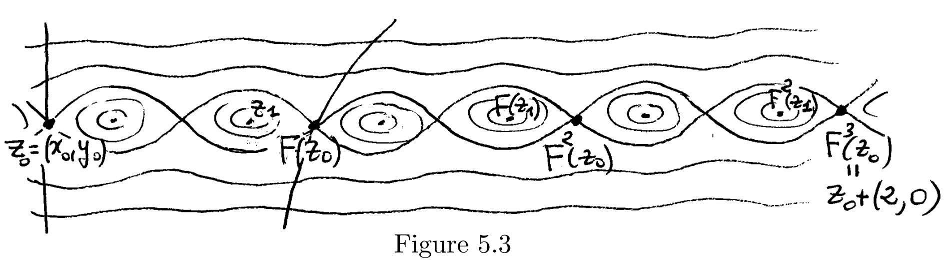


Figure 5.3

In what follows, periodic orbits which are a natural generalization of the "hyperbolic" one will be obtained for a general conservative twist map of the annulus as minima of a certain functional. Generalizations of the "elliptic" one can also be obtained as minimax. Such ideas go back to Birkhoff's works on billards and were developped by Aubry and Le Daeron, Mather and Katok. We shall follow the simple proof given by Katok in [K], which works with a slightly more general definition of the word "conservative" ; indeed, it will be sufficient to suppose that F preserves a measure which is positive on open subsets.
The following definition, in which we follow [K], is directly inspired by this last example (just label the hyperbolic (resp. elliptic) points in natural order):

Definition 31 *Let p, q be relatively prime integers. A Birkhoff point of type (p, q) is a point $z_0 = (x_0, y_0)$ in A whose orbit can be labeled in the following way: there is a sequence $z_n = (x_n, y_n), n \in \mathbb{Z}$, in A, whose projection $x_n, n \in \mathbb{Z}$, on $\mathbb{R}$ is strictly monotone and which satisfies*

$$z_{n+p} = F(z_n), \quad z_{n+q} = z_n + (1, 0).$$

This implies that the projection $\bar{z}_0$ of z_0 on the annulus $\overline{A}$ is a periodic point with rotation number p/q, that no two points of its orbits coincide and that they are ordered as the points in the orbit of the rotation $(x, y) \mapsto (x + p/q, y)$. Such an orbit is the simplest example of a *F-ordered set* as defined below:

Definition 32 *A subset M of A is said to be F-ordered if*
1) M is invariant under F, under F^{-1} and under the integer translations $T_{\pm 1}(x, y) = x \pm 1, y)$;
2) the restriction to M of the projection $\pi(x, y) = x$ is injective;
3) if (x, y) and (x', y') are two elements of M such that $x < x'$, one has $\pi(F(x, y)) < \pi(F(x', y'))$ and $\pi(F^{-1}(x, y)) < \pi(F^{-1}(x', y'))$, where π is the first projection $\mathbb{R}^2 \to \mathbb{R}$.

Being invariant under integer translations, an F-ordered set M projects to an invariant set $\overline{M}$ of $\overline{A}$. In addition to well ordered periodic orbits, basic examples are invariant curves and invariant Cantor sets. A fundamental property of F-ordered sets, whose origin goes back to Birkhoff's works on invariant curves, is stated in the following lemma:

Lemma 33 *Let F be a monotone twist. We suppose that F and F^{-1} are uniformly Lipschitzian. There exists $l > 0$, depending only on F such that, if M is F-ordered and if (x, y) and (x', y') belong to M, one has the uniform Lipshitz estimate*

$$|y - y'| \leq l|x - x'|.$$

Proof. Let us suppose that $y > y'$ (if not, replace F by F^{-1}). The proof can be read on figure 5.4 :

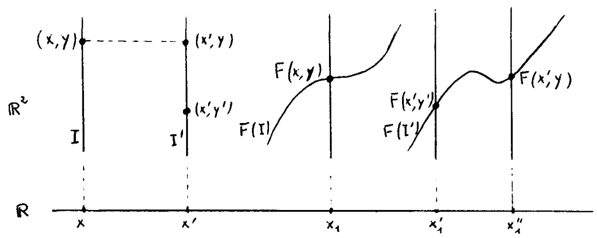


Figure 5.4 : $a(y-y') \le x_1'' - x_1' \le x_1'' - x_1 \le b(x'-x)$.

The first inequality comes from the monotone twist condition, the second one form the fact that M is ordered, and the third one from the fact that F is supposed to be uniformly Lipshitzian. In the perturbative case, when $\overline{F}$ is close to an integrable map, one can get much better estimates for the Lipschitz constant (see [H].

It follows from the Lipschitz estimates that the closure of a F-ordered subset is also F-ordered. The projection $\overline{M}$ on the annulus $\overline{A}$ of a closed F-ordered set $M \subset A$ is called by Katok a *Mather set* (this is not exactly the definition used today but it is convenient to keep it in these notes). The restriction of $\overline{F}$ to a Mather set preserves the cyclic order. Hence, it is topologically conjugate to the restriction to K of some homeomorphism $\overline{f}$ of the circle T^1, where $\overline{K}$ is the (injective) projection of $\overline{M}$ to $\mathbb{T}^1$. This implies that a Mather set has a rotation number. In case of a Birkhoff orbit of type (p,q), this rotation number is obviously equal to $p/q \,(mod\, 1)$.

Every closed subset of a Mather set being itself a Mather set, any Mather set contains a minimal one. The structure of circle homeomorphisms (see [?] Proposition 32 or [L]) that a minimal Mather set is
– either a Birkhoff periodic,
– or an invariant curve on which $\overline{F}$ is conjugated to a rotation with irrational rotation number (which means dense orbits),
– or an invariant Cantor set.

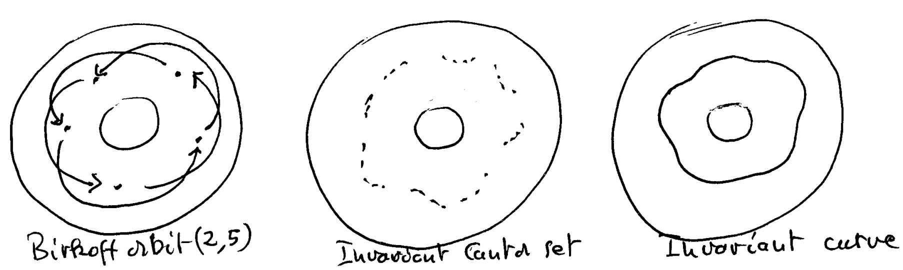


Figure 5.5

Soon after Aubry and Mather had proved the existence of such invariant sets for any rotation number, Katok made the fundamental remark that, because of the Lipschitz estimates, the existence of Mather sets of any irrational rotation number did follow from the existence of Birkhoff periodic orbits. More precisely (see [K] for details):

Proposition 34 *The set of all Mather sets is closed in the Hausdorff topology (defined on the set of all closed subsets of a compact metric space) and the rotation number of a Mather set is continuous in this topology.*

5.2 Existence of Birkhoff periodic orbits: the variational principle

Let $\overline{F} : \overline{A} \to \overline{A}$ be a conservative monotone twist map, $F : A \to A$ is a lift to the covering space. The preservation of orientation and of the measure $dxdy$ implies the preservation of the area 2-form $dx \wedge dy$. If $F(x, y) = (x', y')$, this can be written

$$dx' \wedge dy' = dx \wedge dy,$$

and implies (by the Poincaré lemma) the existence of a function h such that

$$dh(x, x') = -y(x, x')dx + y'(x, x')dx',$$

where $y = y(x, x'$ and $y' = y'(x, x')$ are uniquely defined by the condition that $F(x, y) = (x', y')$ (see the figure below which indicates the obvious interpretation of h). Conversely, h defines F by

$$F\left(x, -\frac{\partial h}{\partial x}(x, x')\right) = \left(x', \frac{\partial h}{\partial x'}(x, x')\right).$$

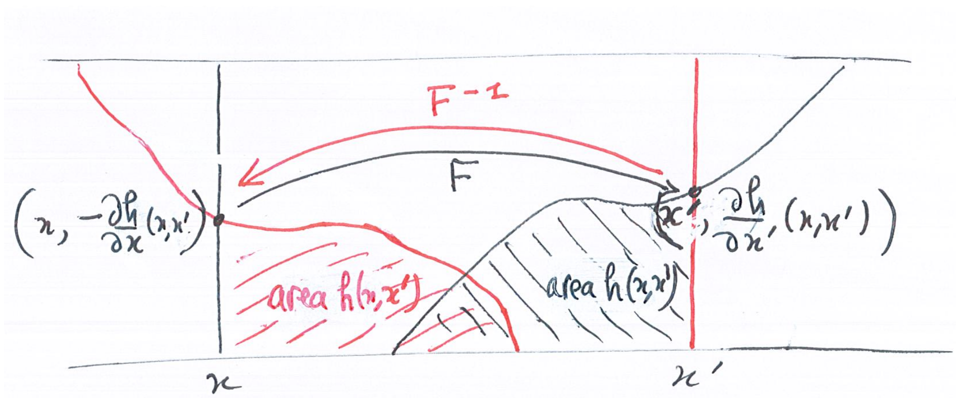


Figure 5.6

Of course, if A is the closed annuuis, h is defined only in the subset B of $\mathbb{R}^2$ defined by

$$B = \{(x, x'),\ f_0(x) \le x' \le f_1(x)\},$$

where f_0 and f_1 are the restrictions of F to the boundaries $\mathbb{R} \times \{0\}$ and $\mathbb{R} \times \{1\}$ of A. Note that h is bounded below and such that $h(x + 1, x' + 1) = h(x, x')$. It is of class at least C^2 and its hessian $\frac{\partial^2}{\partial x \partial x'}$ is everywhere negative.

If p and q are integers, let $X_{p,q}$ be the set of sequences

$$x = (x_i)_{i\in\mathbb{Z}} \quad \text{such that} \quad \forall i \in \mathbb{Z},\ x_{i+q} = x_i + p.$$

The embedding $X_{p,q} \to \mathbb{R}^q$ defined by $x \mapsto (x_0, \cdots, xq-1)$ induces a topology on $X_{p,q}$. Let $W = W_{0,q} : X_{p,q} \to \mathbb{R}$ be defined by

$$W(x) = \sum_{i=0}^{q-1} h(x_i, x_{i+1}).$$

W is invariant under integer translations, i.e. $W(x) = W(T(x))$, where

$$T(x) = (\bar{x_i})_{i\in\mathbb{Z}} \quad \text{with} \quad \bar{x_i} = x_i + 1.$$

The quotient W/T is compact (under our hypotheses this is true for a finite annulus as well as for the infinite cylinder) and W is bounded below, hence it attains its minimum. If the minimum is in the interior of the domain B, it is a critical point, that is: $\frac{\partial W}{\partial x_i} = 0$ for $i = 0, 1, \cdots, q-1$. This implies that

$$\forall i \in \mathbb{Z}, \frac{\partial h}{\partial x} + \frac{\partial h}{\partial x'}(x_i, x_{i+1}) = 0,$$

and hence that $(x_i, -\frac{\partial h}{\partial x}(x_i, x_{i+1})),\ i \in \mathbb{Z}$ is an orbit (see the figure below).

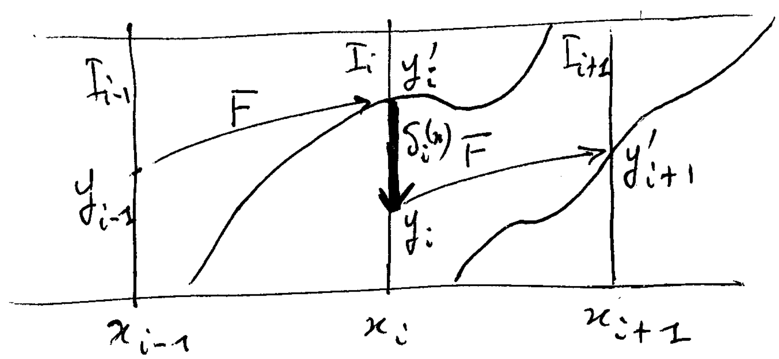


Figure 5.7

In fact, an argument due to Aubry and Le Daeron shows that such an orbitt is necessarily a Birkhoff orbit. All this works nicely in case $\overline{A}$ is the infinite cylinder; in case A is a finite annulus, there are some technical problems due to the existence of a boundary for the domain of definition B of h. We shall explain the proof given by Katok, which solves in a very simple way – indeed without differential calculus – all these problems. As often in mathematics, it will be easier to solve a more general problem, namely the case when the preserved measure is just asked to weight positively each open subset and no regularity beyond continuity is required.

Definition 35 *The interval $[\rho_0, \rho_1]$ defined by the rotation numbers $\rho_i = \rho(F|_{\mathbb{R}\times\{i\}})$ of the restriction of F to the boundary of A is called the twist interval. If $A = \mathbb{R}^2$, the twist interval is $]-\infty, +\infty[$.*

Theorem 36 *Le $\overline{F}$ be a monotone twist homeomorphism of the annulus which preserves a measure $\overline{\mu}$ weighting positively open subsets. Then $\overline{F}$ has a Birkhoff periodic orbit of type (p, q) for any p/q belonging to the twist interval.*

Proof. Influenced by definition 31, we adapt the labeling of sequences to the expected behaviour of the orbit we are looking for: let $M_{p,q}$ be the set of non decreasing bi-infinite sequences $(x_n)_{n\in\mathbb{Z}}$ of real numbers such that, noting $f_i = F|_{\mathbb{R}\times\{i\}}$,

$$x_{n+q} = x_n + 1 \quad \text{and} \quad f_0(x_n) \le x_{n+p} \le f_1(x_n).$$

The topology on $M_{p,q}$ being induced by the embedding $(x_n)_{n\in\mathbb{Z}} \to (x_0, \cdots, x_{q-1})$, its quotient $M_{p,q}/T$ by the integer translations $(x_n)_{n\in\mathbb{Z}} \mapsto (x_n + k)_{n\in\mathbb{Z}}$ is compact. That it is non empty can be seen in the following way: either p/q lies in the interior of the rotation interval and $\forall x$, $f_0^q(x) < x + p < f_1^q(q)$, or it lies on the boundary $\mathbb{R} \times \{i\}$ and $\exists \tilde{x}, \tilde{x} + p = f_i^q(\tilde{x})$ (see [C1] section 4.3, Corollary 27). In the first case, one takes the sequence $x_n = f_t^n(x)$ for some homeomorphism f_t belonging to a monotone family interpolating between f_0 and f_1 and x arbitrary, while in the second case one takes the sequence $f_i^q(\tilde{x})$.
Guided by the case when the lift μ to A of the invariant measure is the Lebesgue measure $dxdy$, we define on $M_{p,q}/T$ the functional

$$W\left((x_n)_{n\in\mathbb{Z}}\right) = \sum_{n=0}^{q-1} \mu\big(\tau(x_n, x_{n+p})\big),$$

where the "triangle" $\tau(x, x')$ is defined on the figure below:

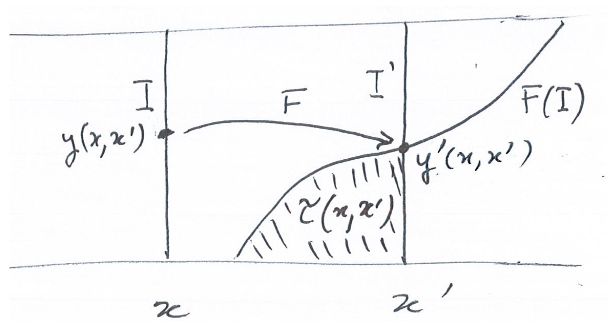


Figure 5.8

The claim is that any local minimum of W is a Birkhoff orbit of type (p, q). As such an orbit satisfies $F(z_n) = z_{n+p}$, it suffices to prove, as already explained (with different notations for the sequences), that at a local minimum of W, one has

$$\forall n \in \mathbb{Z}, \quad y(x_n, x_{n+p}) = y'(x_{n-p}, x_n),$$

where we recall that, if $(x, x') \in B$, $y(x, x')$ and $y'(x, x')$ are uniquely defined by the equality $F(x, y(x, x')) = (x', y'(x, x'))$.

The proof of this equality is by showing that if it is not satisfied for some i, there exists a small perturbation of the sequence, which lowers W. The different cases are illustrated in the following figures.

We suppose first that $x_{n-1} < x_n < x_{n+1}$ and $y'(x_{n-p}, x_n) > y(x_n, x_{n+p})$ (left) or $y'(x_{n-p}, x_n) > y(x_n, x_{n+p})$ (right). Moving a little x_n to the left or to the right we see that the preservation of μ leads to a contradiction: indeed, in both cases, the sum $\mu\big(\tau(x_{n-p}, x_n)\big) + \mu\big(\tau(x_n, x_{n+p})\big)$ has decreased; in the first case this is because the increase of $\mu\big(\tau(x_n, x_{n+p})\big)$ is smaller than the decrease of $\mu\big(\tau(x_{n-p}, x_n)\big)$, while in the second one, the decrease of $\mu\big(\tau(x_n, x_{n+p})\big)$ is greater than the increase of $\mu\big(\tau(x_{n-p}, x_n)\big)$.

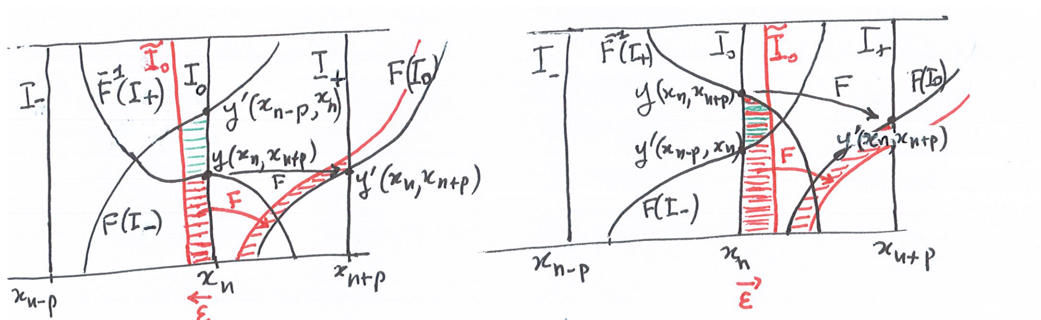


Figure 5.9

Now suppose more generally that $x_{n-1} < x_n = x_{n+1} = \cdots = x_{n+k} < x_{n+k+1}$. The twist property implies

$$\begin{aligned} 1 \geq y'(x_{n-p}, x_n) \geq \cdots \geq y'(x_{n-p+k}, x_{n+k}) \geq 0, \\ 1 \geq y(x_{n+p+k}, x_{n+k}) \geq \cdots \geq y(x_{n+p}, x_n) \geq 0, \end{aligned}$$

hence either $y'(x_{n-p}, x_n) \geq y(x_{n+p}, x_n)$ or $y(x_{n+p+k}, x_{n+k}) \geq y'(x_{n-p+k}, x_{n+k})$, whichis similar to the first case, or for some l between 0 and k, $y(x_{n+l}, x_{n+p+l}) = y'(x_{n-p+l}, x_{n+l})$.

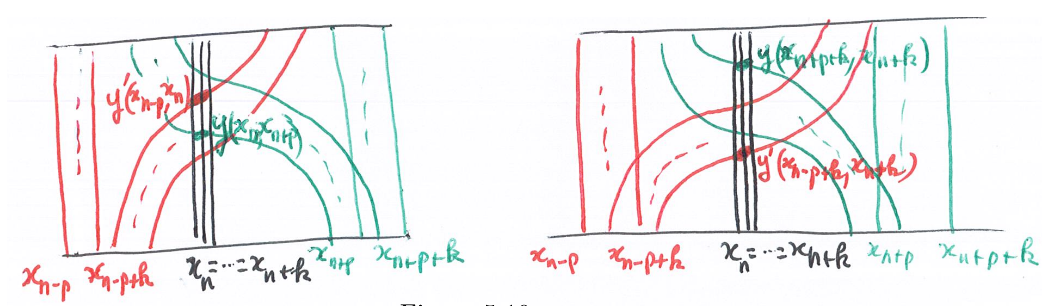


Figure 5.10

Note that, in contrast with the use of differential calculus, the cases when some $y(x_i, x_{i+p})$ or $y'(x_{i-p}, x_i)$ belongs to the boundary have nothing special.

5.3 Readings

1) [M2] J. Mather, *Non-existence of invariant circles*, In this short paper, John Mather studies monotone twist mappings of the open cylinder $\mathbb{T}^1 \times \mathbb{R}$ of the form

$$(x, y) \mapsto (x' = x + y + h(x),\ y' = y + h(x)), \quad \text{with} \quad \int_0^1 h(x)dx = 0.$$

The condition on the integral of h is easily seen to be necessary for the existence of an invariant curve homotopic to the circles $\mathbb{T}^1 \times a$. By a very simple proof relying on Birkhoff's Lipschitz estimates for such an invariant curve, Mather shows that for Chrikov's *standard map*, which is the case when $h(x) = \frac{k}{2\pi} \sin 2\pi x$, no such invariant curve exists for $k > 4/3$.

2) [Mo1] J. Moser *Monotone Twist Mappings and the Calculus of Variations*, Here, Jurgen Moser proves that any C^∞ twist map may be considered as the Poincaré return map of a time periodic Hamiltonian in the sense of section 4.2. The converse is no true, what plays the role of Hamiltonians satisfying the Legendre condition $\frac{\partial L}{\partial \dot{q}^2}$ being compositions of onotone twist maps.

6 The Poincaré-Birkhoff-Conley twist map of the annulus for the planar restricted 3-body problem

The problem we study in this section is a "caricature" of the 3-body problem (Sun, Earth Moon or, more accurately, Sun, Jupiter and a satellite), which shares many properties, as Poincaré had already noticed, with the geodesic flow on a convex surface endowed with the induced metric of the euclidean space $\mathbb{R}^3$. It is a historicaly fundamental example of a monotone twist map of the annulus.

6.1 The Kepler problem as an oscillator

The (normalized) motions in a plane of a particle submitted to the Newtonian attraction of a fixed center – the so called *Kepler problem* – are the solutions of the equation

$$\ddot{x} = -x/|x|^3,$$

where $x \in \mathbb{R}^2 = \mathbb{C}$ is identified with a complex number and the dot denotes the time derivative. These equations are the Hamilton equations

$$\dot{x} = \frac{\partial H}{\partial \bar{y}},\ \dot{y} = -\frac{\partial H}{\partial \bar{x}}$$

associated to the Hamiltonian $H : (\mathbb{C} \setminus \{0\}) \times \mathbb{C} \to \mathbb{R}$ and the symplectic form ω respectively defined by

$$H(x,y) = |y|^2 - 2/|x|, \quad \omega = dx \wedge d\bar{y} + d\bar{x} \wedge dy.$$

Exercise. Show that, if $x = x_1 + ix_2$ and $y = y_1 + iy_2$, these are, up to a factor 2, the classical Hamilton equations and symplectic form:

$$\dot{x}_i = \frac{1}{2}\frac{\partial H}{\partial y_i},\ \dot{y}_i = -\frac{1}{2}\frac{\partial H}{\partial x_i}, \quad i = 1,2, \quad \omega = 2(dx_1 \wedge dy_1 + dx_2 \wedge dy_2).$$

The Levi-Civita mapping $(z,w) \mapsto \big(x = 2z^2,\ y = w/\epsilon\bar{z}\big)$ defines a two-fold covering

$$(L.C.) \qquad K_2^{-1}(0) \setminus \{z=0\} \to \Sigma_\epsilon = H^{-1}(-1/\epsilon^2)$$

from the complement $\tilde{\Sigma}_\epsilon$ of the plane $z = 0$ in the 0-energy 3-sphere $K_2^{-1}(0)$ of the harmonic oscillator

$$K_2(z,w) = |z|^2 + |w|^2 - \epsilon^2 = \epsilon^2|z|^2\left[H\left(2z^2, w/\epsilon\bar{z}\right) + 1/\epsilon^2\right],$$

to the energy hypersurface $\Sigma_\epsilon = H^{-1}(-1/\epsilon^2)$ of the Kepler problem (both diffeomorphic to $S^1 \times \mathbb{R}^2$).

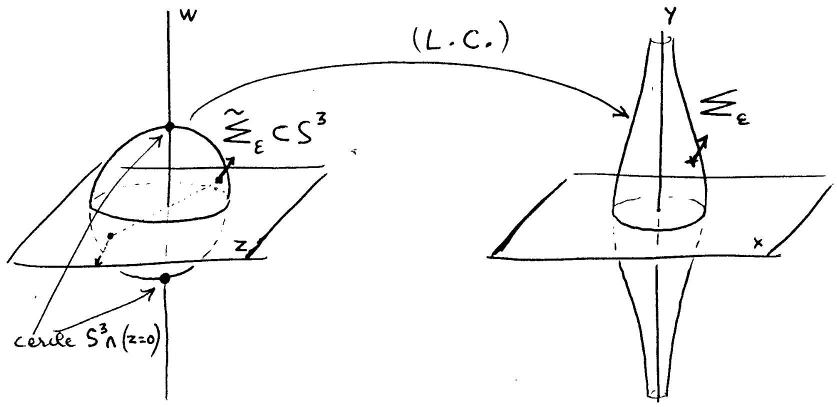


Figure 6.1: The Levi-Civita map

It is conformally symplectic (precisely $K_2^*\omega = \frac{4}{\epsilon}(dz \wedge d\bar{w} + d\bar{z} \wedge dw)$) and sends integral curves of the harmonic oscillator with energy ϵ^2 to those of the Kepler problem with energy $-1/\epsilon^2$ after the change of time $dt = 2\epsilon|x|dt'$ which prevents the velocity to become infinite at collision.
In the still conformally symplectic coordinates

$$u_1 = w + iz, \quad u_2 = \bar{w} + i\bar{z}, \quad du_1 \wedge d\bar{u}_1 + du_2 \wedge d\bar{u}_2 = 2i(dz \wedge d\bar{w} + d\bar{z} \wedge dw)$$

these integral curves are $u_1(t) = c_1 e^{it},\ u_2(t) = c_2 e^{it},\ |c_1|^2 + |c_2|^2 = 2\epsilon^2$, that is the intersections of the 3-sphere with the complex lines $u_1/u_2 = cste$, or in other words the fibers of the *Hopf fibration* $(u_1, u_2) \mapsto u_1/u_2 : S^3 \to P_1(\mathbb{C}) = S^2$. The closest approximation to a (bivalued) section of the Hopf map, the annulus

$$\arg u_1 + \arg u_2 = 0 \pmod{2\pi}$$

is a (bivalued) global surface of section of the flow of the Harmonic oscillator in a sphere of constant energy: with the exception of the two fibers which form its boundary, all the fibers cut this annulus transversally in two points; hence, the second return map is the identity. Thus, perturbations of the Kepler problem with negative energy are essentially perturbations of the identity map. This is one of the main sources of degeneracies in celestial mechanics.

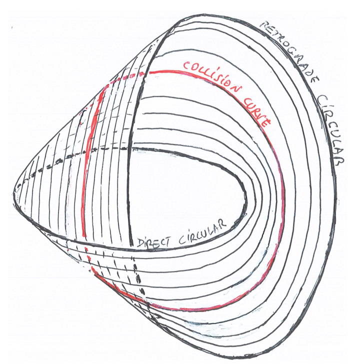


Figure 6.2: the annulus of section

Remarks. 1) We kept the two-fold covering because working in $S^3 \subset \mathbb{R}^4$ is convenient. To get rid of the bivaluedness of the section it suffices to go to the quotient by the antipody, replacing S^3 by $\mathbb{R}P^3 = SO(3)$. The return map then becomes truly the identity.

2) Show that $\frac{1}{2\epsilon}\left(|u_1|^2 - |u_2|^2\right) = \frac{1}{i\epsilon}(\bar{z}w - z\bar{w}) = x_1y_2 - x_2y_1$ is the *angular momentum* of the motion defined by the element (x, y) of the phase space.

Exercise: retrieving the Kepler ellipses. From the solution of the equations in (u_1, u_2) space,

$$u_1(t) = r_1 e^{i(s_1+t)}, \quad , u_2(t) = r_1 e^{i(s_2+t)},$$

show that the solutions i (x, y) space lie on ellipses (recall that the total energy $-\frac{1}{\epsilon^2}$ is negative) given by the following equations:

$$x = -\frac{1}{2}e^{i\varphi}\left[(r_1^2 + r_2^2)\cos\chi - 2r_1r_2 + i(r_1^2 - r_2^2)\sin\chi\right],$$

where the angles φ, χ are well defined mod π by the formulas

$$\varphi = s_1 - s_2 = \operatorname{Arg} u_1 - \operatorname{Arg} u_2, \quad \chi = s_1 + s_2 + 2t = \operatorname{Arg} u_1 + \operatorname{Arg} u_2.$$

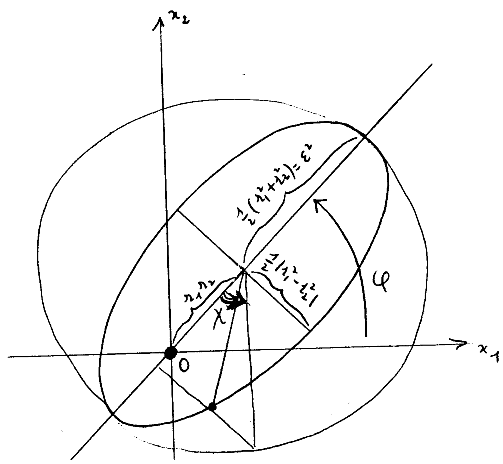


Figure 6.3: A Kepler ellipse

6.2 The restricted problem in the lunar case

The equations of the n-body problem

$$\frac{d^2\vec{r}_i}{dt^2} = g\sum_{j\neq i}\frac{m_j(\vec{r}_j - \vec{r}_i)}{||\vec{r}_i - \vec{r}_j||^3}$$

make sense even if some of the masses vanish. Such masses are influenced by the non-zero masses but do not influence them. We shall consider two primaries, say the Sun (mass μ) and the Earth (mass ν) which have a uniform circular motion around their center of mass and a 0-mass third body, say the Moon, which stays close to the Earth. We identify the inertial plane with $\mathbb{C}$ (coordinate $X = X_1 + iX_2$ centered on the center of mass of the couple Sun-Earth) and introduce rotating *(synodic)* complex coordinates (ζ, u) by setting

$$X = \zeta e^{i\omega t}, \quad Y = \dot{X} = u e^{i\omega t}, \quad \text{that is} \quad u = \dot{\zeta} + i\omega\zeta.$$

The equations become

$$\ddot{\zeta} + 2i\omega\dot{\zeta} - \omega^2\zeta = g\mu\frac{\zeta_S - \zeta}{|\zeta_S - \zeta|^3} + g\nu\frac{\zeta_E - \zeta}{|\zeta_E - \zeta|^3},$$

where $\zeta_S = -\frac{\nu}{\mu+\nu}r_0$ and $\zeta_E = \frac{\mu}{\mu+\nu}r_0$ are the respective (fixed) positions of the Sun and the Earth in the rotating frame. They take the following Hamiltonian form (independent of t because of rotational invariance):

$$\frac{d\zeta}{dt} = \frac{\partial H_{syn}}{\partial \bar{u}}, \quad \frac{du}{dt} = -\frac{\partial H_{syn}}{\partial \bar{\zeta}}, \quad \text{where}$$

the Hamiltonian and the symplectic form are respectively

$$\begin{cases} H_{syn}(\zeta, u) = |u|^2 + 2\omega \mathrm{Im}(\zeta\bar{u}) - 2\dfrac{g\mu}{|\zeta_S - \zeta|} - 2\dfrac{g\nu}{|\zeta_E - \zeta|}, \\ \qquad \omega_{syn} = du \wedge d\bar{\zeta} + d\bar{u} \wedge d\zeta. \end{cases}$$

Due to the invariance under rotation of the problem, H_{syn} is independent of time. We shall use slightly different coordinates, centered on the earth:

$$x = \zeta - \frac{\mu}{\mu+\nu}r_0, \quad y = u - i\omega\frac{\mu}{\mu+\nu}r_0.$$

Moreover we shall normalize the equations by setting

$$g = 1, \quad \mu + \nu = 1, \quad r_0 = 1, \quad \text{so that} \quad \omega = \sqrt{g(\mu+\nu)}/r_0^{\frac{3}{2}} = 1.$$

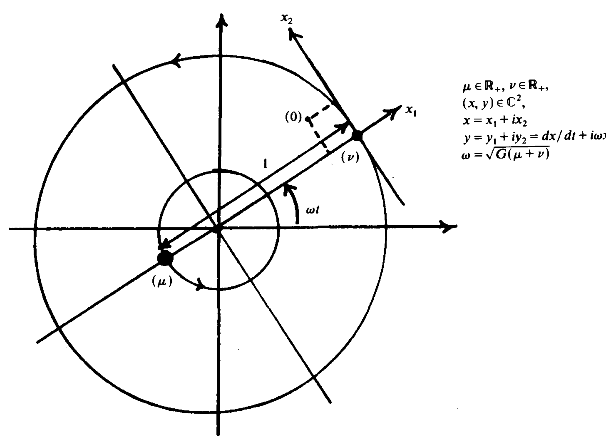


Figure 6.4: Rotating coordinates

The equations of motion of the Moon become

$$\dot{x} = \frac{\partial H}{\partial \bar{y}}, \;\; \dot{y} = -\frac{\partial H}{\partial \bar{x}},$$

where the Hamiltonian (up to the constant term which we have changed) and the symplectic form are respectively

$$\begin{cases} H(x,y) = |y|^2 + i\omega(\bar{x}y - x\bar{y}) - \dfrac{2\nu}{|x|} - \dfrac{2\mu}{|x+1|} - \mu(x+\bar{x}) + 2\mu, \\ \omega = dy \wedge d\bar{x} + d\bar{y} \wedge dx = 2(dy_1 \wedge dx_1 + dy_2 \wedge dx_2). \end{cases}$$

As in the first section, we consider the energy hypersurface $H^{-1}(1/\epsilon^2)$, with ϵ a small parameter. Its projection on the x plane is made of three connected components: a neighborhood of the Sun, a neighborhood of the Earth and a neighborhood of infinity (the so-called Hill's regions, which imply Hill's stability result, praised by Poincaré).

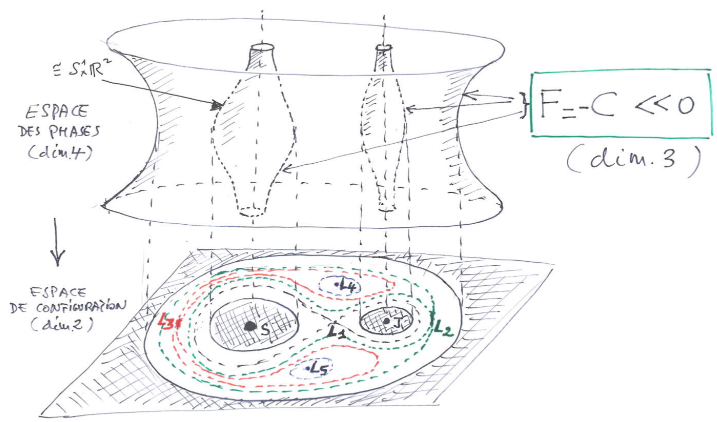


Figure 6.5: Hill's regions

We shall be interested in the connected component of $H^{-1}(1/\epsilon^2)$ where $|x|$ stays small. Then

$$H(x,y) = |y|^2 + i\omega(\bar{x}y - x\bar{y}) - \frac{2\nu}{|x|} - 2\mu\left[\frac{1}{4}|x|^2 + \frac{3}{8}(x^2 + \bar{x}^2) + O_3(x)\right].$$

We see that the influence of the Sun on the Moon becomes negligible with respect to the one of the Earth and that at the collision limit, it disappears and one is left with a Kepler problem. To make this apparent, we again apply the Levi-Civita transformation $(z,w) \mapsto \big(x = 2z^2,\; y = w/\epsilon\bar{z}\big)$. We get

$$K(z,w) = \epsilon^2|z|^2\left[H\left(2z^2, \frac{w}{\epsilon\bar{z}}\right) + \frac{1}{\epsilon^2}\right] = f^2(z,w)|z|^2 + |w|^2 - \nu\epsilon^2 - \epsilon^2\mu g(z),$$

where

$$f(z,w)=\sqrt{1+2i\epsilon(\bar{z}w-z\bar{w})},\quad g(z)=2|z|^2\left(\frac{1}{|2z^2+1|}-1+z^2+\bar{z}^2\right).$$

As in the Kepler case, the direct image of the restriction to $K^{-1}(0)\setminus\{z=0\}$ of the Hamiltonian flow $\dot{z}=\frac{\partial K}{\partial\bar{w}}$, $\dot{w}=-\frac{\partial K}{\partial\bar{z}}$ becomes the flow of the restricted problem with Jacobi constant $-1/\epsilon^2$ after the change of time $dt=2\epsilon|x|dt'$.

Each truncation of the Taylor expansion of $K(z,w)$ at the origin,

$$K(z,w)=-\nu\epsilon^2+|z|^2+|w|^2+2i\epsilon|z|^2(\bar{z}w-\bar{w}z)-\epsilon^2\mu(2|z|^6+3|z|^2(z^4+\bar{z}^4)+0_8(z)),$$

makes sense dynamically when restricted to $K^{-1}(0)$: we get

at order 2, the harmonic oscillator, which regularizes the Kepler problem;
at order 4, the regularization of the Kepler problem in a rotating frame, which provides a nice example of a monotone twist map.
at order 6, *Hill's problem*, one of the simplest "non integrable" problems.

6.3 The regularized Kepler problem in a rotating frame: annulus of section and return map

The truncation at fourth order $K_4(z,w)=-\nu\epsilon^2+f^2(z,w)|z|^2+w^2$ of K, which amounts to equating the mass of the Sun to zero, is a completely integrable Hamiltonian, a first integral being the angular momentum or, what is equivalent, the function $f^2(z,w)$. This is not surprising as we know that the restriction to $K_4^{-1}(0)$ is the regularization of the Kepler problem in a rotating frame, the solutions of which are all periodic or quasi periodic: the intersection of level hypersurfaces of K_4 and f^2 defines in general a two-dimensional torus, except when the two hypersurfaces are tangent, that is when $w=\pm if(z,w)z$. In this case the intersection degenerates to a circle; in $K_4^{-1}(0)$, this defines two solutions which project (by a 2-1 map) onto the two circular solutions (one direct, one retrograde) of the rotating Kepler problem with the given value $-1/\epsilon^2$ of the Jacobi constant.

We set

$$\xi_1=w+if(z,w)z,\quad \xi_2=\bar{w}+if(z,w)\bar{z},$$

which turns the circular solutions into $\xi_i=0,\ i=1,2$. Note that this is not a symplectic change of coordinates. The equations of motion become

$$\begin{cases}\dfrac{d\xi_1}{dt}=i\xi_1\left(\tilde{f}\big(|\xi_1|^2-|\xi_2|^2\big)-\dfrac{\epsilon}{2\tilde{f}^2\big(|\xi_1|^2-|\xi_2|^2\big)}|\xi_1-\bar{\xi}_2|^2\right),\\[2ex] \dfrac{d\xi_2}{dt}=i\xi_2\left(\tilde{f}\big(|\xi_1|^2-|\xi_2|^2\big)+\dfrac{\epsilon}{2\tilde{f}^2\big(|\xi_1|^2-|\xi_2|^2\big)}|\xi_1-\bar{\xi}_2|^2\right),\end{cases}$$

where the function $\tilde{f}$ of one real variable X is defined implicitly by

$$\tilde{f}(X)-\tilde{f}^3(X)=\epsilon X,\quad\text{that is}\quad \tilde{f}(X)=1-\frac{\epsilon}{2}X+O(\epsilon^2X^2).$$

We set

$$\xi_1 = r_1 e^{i\alpha_1}, \quad \xi_2 = r_2 e^{i\alpha_2}.$$

Obviously, $r_1^2 = |\xi_1|^2$ and $r_2^2 = |\xi_2|^2$ are constants of motion as well as

$$\frac{1}{2}\big(r_1^2 + r_2^2\big) - \nu\epsilon^2 = K_4(z,w) \quad \text{and} \quad \epsilon\big(r_1^2 - r_2^2\big) = f(z,w) - f^3(z,w).$$

One checks readily that, writing $\tilde{f}$ for $\tilde{f}(r_1^2 - r_2^2)$,

$$\begin{cases} \dfrac{dr_1}{dt} = 0, \quad \dfrac{dr_2}{dt} = 0, \\ \dfrac{d\alpha_1}{dt} = 1 - \epsilon\dfrac{1}{2\tilde{f}^3}\big[r_1^2 + r_2^2 - 2r_1r_2\cos(\alpha_1+\alpha_2)\big], \\ \dfrac{d\alpha_2}{dt} = 1 + \epsilon\dfrac{1}{2\tilde{f}^3}\big[r_1^2 + r_2^2 - 2r_1r_2\cos(\alpha_1+\alpha_2)\big], \end{cases}$$

which, implying $\frac{d\alpha_1}{dt} + \frac{d\alpha_2}{dt} = 2$, are easily integrated.

$$\begin{cases} r_1(t) = r_1, \quad r_2(t) = r_2, \\ \alpha_1(t) = \alpha_1(0) + t - \epsilon\dfrac{(r_1^2 + r_2^2)}{2\tilde{f}^3}t + \epsilon\dfrac{r_1r_2}{2\tilde{f}^3}\left[\sin\big(\alpha_1(0) + \alpha_2(0) + 2t\big) - \sin\big(\alpha_1(0) + \alpha_2(0)\big)\right], \\ \alpha_2(t) = \alpha_1(0) + t + \epsilon\dfrac{(r_1^2 + r_2^2)}{2\tilde{f}^3}t - \epsilon\dfrac{r_1r_2}{2\tilde{f}^3}\left[\sin\big(\alpha_1(0) + \alpha_2(0) + 2t\big) - \sin\big(\alpha_1(0) + \alpha_2(0)\big)\right]. \end{cases}$$

The annulus twist map : The same annulus

$$\arg\xi_1 + \arg\xi_2 = 0 \pmod{2\pi}$$

as the one we used for the Kepler problem in the inertial frame will capture the global dynamics, forgetting only the time law. Note that it contains the collision circle $z = 0$ (that is $\xi_1 - \bar{\xi}_2 = 0$). It is bounded by the two periodic solutions $\xi_1 = 0$ and $\xi_2 = 0$ which, corresponding to the circular solutions, are geometrically indifferent to the rotation of the frame.

As $(\alpha_1 + \alpha_2)(t) = (\alpha_1 + \alpha_2)(0) + 2t$, the first return map coincides with the "time π" map of the flow; it amounts to picking on each orbit the aphelium (point of the moon trajectory closest to the Earth), a point which moves on a circle in the direction opposite to the rotation of the frame.

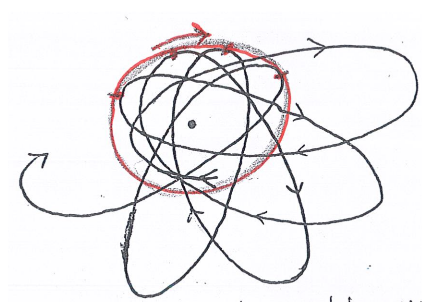


Figure 6.6: Rotating ellipses

Taking $\theta = \alpha_1$ and $r = r_1^2 - r_2^2$ as coordinates in the interior of the annulus (which is then bounded by the curves $r = \pm 2\nu\epsilon^2$), the first return map on the annulus is

$$P(\theta, r) = \left(\theta + \frac{1}{2} - \frac{\nu}{2\tilde{f}^3(r)}\epsilon^3, r\right) = \left(\theta + \frac{1}{2} - \frac{\nu}{2}\epsilon^3 - \frac{3\nu}{4}\epsilon^4 + h.o.t., r\right).$$

Finally, setting $\rho = 2\nu\epsilon^2 r$ so that the equation of the boundary of the annulus becomes $\rho = \pm 1$, we get (keeping by a slight abuse the same notation P)

$$P(\theta, \rho) = \left(\theta + \frac{1}{2} - \frac{\nu}{2}\epsilon^3 - \frac{3\nu^2}{2}\epsilon^6\rho + 0(\epsilon^8),\ \rho\right),$$

which is as expected an integrable monotone twist of the standard annulus which preserves a measure regular with respect ot the Lebesgue measure. Of course, with a little extrawork one could have chosen a symplectic change of coordinates and keep the standard measure.

6.4 A glimpse of the rest of the story

One can show (see [C4] and the references it contains) that when the perturbation of the Sun is added, the circular orbits can be continued to almost circular periodic orbits which can serve as boundaries of a global annulus of section of the energy manifold (still diffeomorphic to a 3-sphere) of the regularized problem. After some computing, one finds a return map of the following form in an annulus whose boundaries are close to $\rho = \pm 1$ and which one can chose to contain the collision curve:

$$P_\epsilon(\theta, \rho) = \left(\theta + \frac{1}{2} - \frac{\nu}{2}\epsilon^3 - \frac{3\nu^2}{2}(1 - \frac{\mu}{4})\epsilon^6\rho + 0(\epsilon^7),\ \rho + O(\epsilon^7)\right).$$

Coming back to the definition of this annulus, one checks that the return map corresponds essentially to the passages of the orbit of the Moon through aphelium in the rotating frame. Originating from a Hamiltonian system, this map necessarily preserves a measure defined by a smooth density.

As it is a $O(\epsilon^7)$ perturbation of an integrable twist map whose twist is of size ϵ^6. This is a perfect ground for applying the main results of the general theory of conservative twist maps, a particular case of the theory of Hamiltonian systems with two degrees of freedom:
1) Applied to the iterates of the return map, the *Birkhoff fixed point theorem* yelds an infinite number of periodic orbits of higher and higher periods to which correspond periodic orbits of long period of the Moon around the Earth in the rotating frame;
2) The *Moser invariant curve theorem* implies the existence of a positive measure Cantor set of invariant curves on which the map is conjugated to a diophantine irrational rotation and to which correspond quasi periodic orbits of the Moon;

3) To the Liouville rotation numbers, the *Aubry-Mather theory* associates invariant Cantor sets to which correspond orbits of the Moon with a Cantor caustic

4) Finally, it is possible to prove that the image of the collision circle intersects itself transversally at eight points [**?**]; in particular, it is not contained in an invariant curve. Varying the value of ϵ moves the invariant curve of a given rotation number across the annulus which forces intersection with the collision curve. This proves the existence of invariant "punctured" tori which correspond to orbits of the Moon which persistently change their direction of rotation around the Earth in the rotating frame.

Remark. For writing down formulas, working in the 2-fold covering $K^{-1}(0)$ of the energy hypersurface diffeomorphic to S^3 is convenient but one can prefer to state the results downstairs in the compactification (regularization), diffeomorphic to $SO(3)$ (that is to the real projective space of dimension 3), of the original energy hypersurface $H^{-1}(-\frac{1}{\epsilon^2})$. The first return map then becomes a perturbation of the Identity (the Kepler case) of the form

$$\mathcal{P}(\tilde{\theta},\rho) = \Big(\tilde{\theta} - \nu\epsilon^3 - 3\nu^2(1-\frac{\mu}{4})\epsilon^6\rho + 0(\epsilon^7),\, \rho + O(\epsilon^7)\Big).$$

On Figure 7-7 are summarized some features of the complicated dynamics of the return map of the restricted three-body problem in the lunar case (see [C2, C4]; the roman numbers refer to Chapters of Poincaré's *New Methods of Celestial Mechanics*).

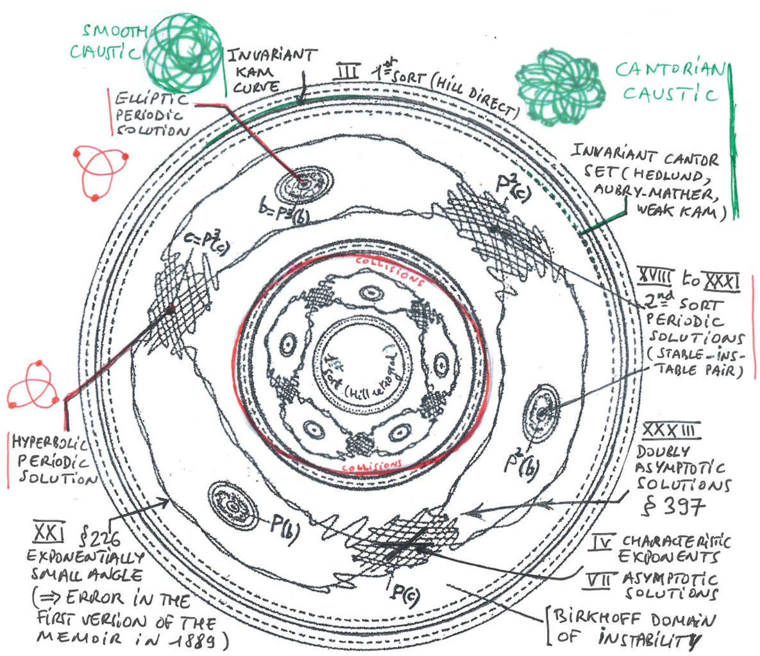


Figure 6.7

7 Appendix: a brief introduction to differential forms

We deal essentially with the (local) case of an open subset Ω of $\mathbb{R}^n$. The definitions are given in such a way that the generalization to the (global) case of a manifold M is straightforward.

7.1 Tangent and cotangent bundle

Given an open subset Ω of $\mathbb{R}^n$, the tangent space $T_{x_0}\Omega$ and the cotangent space $T^*_{x_0}\Omega$ at a point $x_0 \in \Omega$ may be intrinsically defined as follows: $T_{x_0}\Omega$ is the quotient of the set of C^1 local paths $c : (\mathbb{R}, 0) \to (\Omega, x_0)$ by the equivalence relation which identifies two local paths c_1 and c_2 if $(c_1 - c_2)(t) = o(t)$. The equivalence class is the velocity vector $c_1'(0) = c_2'(0)$.

In the same way, $T^*_{x_0}\Omega$ is the quotient of the set of C^1 local maps $f : (\Omega, x_0) \to (\mathbb{R}, 0)$ by the equivalence relation which identifies two local maps f_1 and f_2 if $(f_1 - f_2)(x) = o(|x - x_0|)$. The equivalence class is the derivative $df_1(x_0) = df_2(x_0)$. Moreover, the coupling induced by $(f, c) \mapsto (f \circ c)'(0)$ identifies naturally $T^*_{x_0}\Omega$ with the dual of $T_{x_0}\Omega$.

One checks that the addition of maps endows $T_{x_0}\Omega$ and $T^*_{x_0}\Omega$ with the structure of real vector spaces.

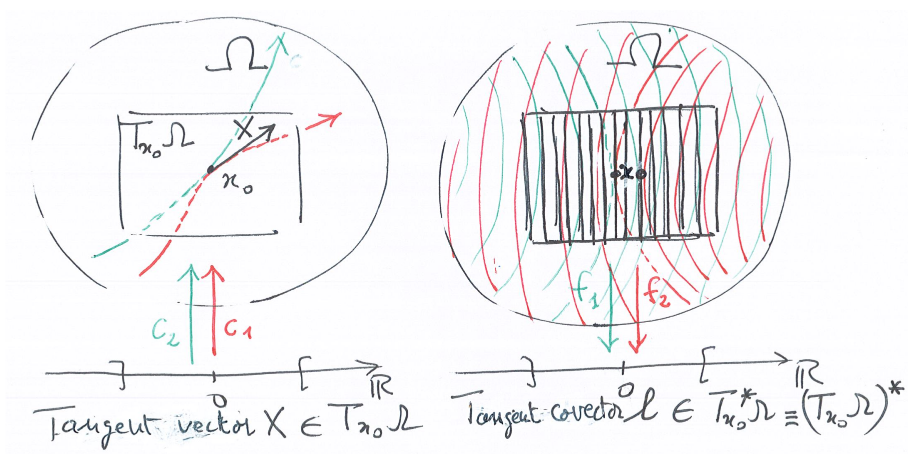


Figure 7.1

These definitions generalize immediately to the case when Ω is replaced by a manifold M.

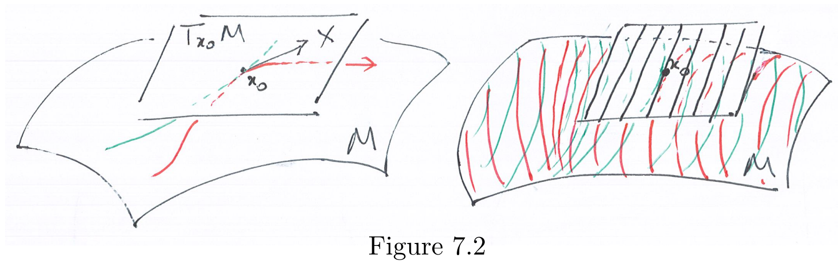


Figure 7.2

What does not generalize from the case of Ω to the one of M is the fact that, due to the existence of translations in $\mathbb{R}^n$, there are canonical identifications of $T_{x_0}\Omega$ with $T_0\mathbb{R}^n \equiv \mathbb{R}^n$ and of $T^*_{x_0}\Omega$ with $T^*_0\mathbb{R}^n \equiv (\mathbb{R}^n)^*$. This provides canonical identifications of the disjoint union $T\Omega$ (resp. $T^*\Omega$) of tangent (resp. cotangent) spaces at all points of Ω to the product $\Omega \times \mathbb{R}^n$ (resp. $\Omega \times (\mathbb{R}^n)^*$).

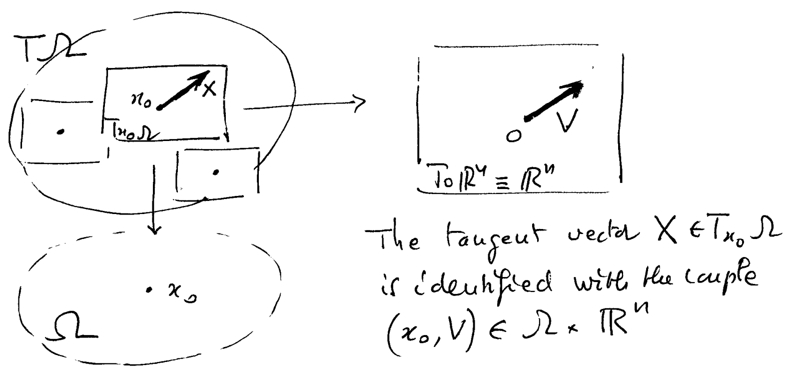


Figure 7.3

On a manifold, such a (non canonical) product structure exists only locally and provides TM (resp. T^*M) with the structure of a vector bundle over M.

Given a differentiable map F from a manifold M to another manifold N, its tangent map $TF : TM \to TN$ is uniquely defined as the map which sends the velocity vector $X_1 = c_1'(0) \in T_{x_1}M_1$ of a path c_1 in Ω_1 such that $c_1(0) = x_1$ to the velocity vector $X_2 = c_2'(0) \in T_{x_2}M_2$ of the path $c_2 = F \circ c_1$ in M_2. In the case $M_1 = \Omega_1$ and $m_2 = \Omega_2$ are open subsets respectively of $\mathbb{R}^{n_1}$ and $\mathbb{R}^{n_2}$, after making the canonical identifications $T_{x_1}\Omega_1 \equiv \mathbb{R}^{n_1}$ and $T_{x_2}\Omega_2 \equiv \mathbb{R}^{n_2}$, one gets back the elementary definition of the derivative $dF(x_1)$ as a linear map from $\mathbb{R}^{n_1}$ to $\mathbb{R}^{n_2}$.

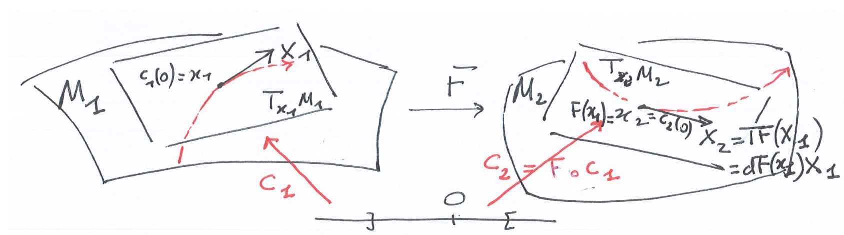


Figure 7.4

Note that the consideration of the tangent map simplifies the expression of the derivative of a composed map which becomes simply

$$T(G \circ F) = TG \circ TF.$$

7.2 Vector fields and 1-forms (= covector fields)

Associating differentiably to each point $x \in \Omega$ a tangent vector $X(x) \in T_x\Omega$ (resp. a tangent covector $l(x) \in T_x^*\Omega$) one defines a differentiable vector field (resp. covector field) on Ω. Through the canonical indentification of $T_x\Omega$ with $\mathbb{R}^n$, a vector field on Ω becomes a differentiable map from Ω to $\mathbb{R}^n$ and a covector field (or 1-form) becomes a differentiable map from Ω to $(\mathbb{R}^n)^*$.

In the case of a manifold, a vector field on M is a section of the vector bundle $TM \to M$, that is a map $\sigma : M \to TM$ such that $\pi \circ \sigma =$ Identity. In the same way a covector field (aso called a *differential 1-form*, or simply *1-form*) is a section of the vector bundle T^*M.

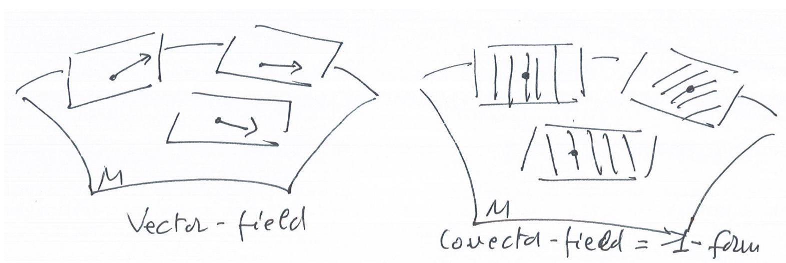


Figure 7.5

The first example of differential 1-form is the derivative of a function $F : M \to \mathbb{R}$. Indeed, at each point $x \in M$ one associates the derivative $dF(x)$ which is a linear map from T_xM to $T_{F(x)}\mathbb{R} \equiv \mathbb{R}$, that is an element of T_x^*M. Considered as a differential 1-form on M it is noted dF.

In case $M = \Omega \subset \mathbb{R}^n$, one has the canonical decomposition $dF = \sum_{i=1}^n \frac{\partial F}{\partial x_i} dx_i$, where the dx_i are the derivatives of the coordinate functions $x = (x_1, \cdots, x_n) \mapsto x_i,\ i = 1, \cdots n$. At each point $x \in \Omega$, $\{dx_1(x), \cdots, dx_n(x)\}$ is the basis of $(\mathbb{R}^n)^* \equiv T_x^*\Omega$ dual to the canonical basis of $\mathbb{R}^n$.

One of the advantages of 1-forms on vector fields can be seen when comparing the natural operations on them: direct image (or push forward) F_*X of a vector field X on M by a differentiable map $F : M \to N$, inverse image (or pull back) $F^*\omega$ of a 1-form ω on N by the map F. The first one can be defined only when F is a diffeomorphism while the second is defined for any differentiable map F :

$$(F_*X)(y) = dF(F^{-1}(y))X(F^{-1}(y)) \in T_yN, \quad (F^*\omega)(x) = \omega(F(x)) \circ dF(x) \in T_x^*M.$$

Note that the derivation of composed functions takes the nice form

$$d(G \circ F) = F^*dG,$$

and that, if F is a diffeomorphism and X a vector field on M, one has

$$< (F^*\omega), X >=< \omega, F_*X > .$$

The great simplicity of computations with differential forms appears clearly on the formulas in coordinates when $F = (F_1, \cdots, F_p) : \Omega \to \mathcal{O}$ is a map from an open set $\Omega \subset \mathbb{R}^n$ (coordinates $x_1, \cdots, x_n$) to an open set $\mathcal{O} \subset \mathbb{R}^p$ (coordinates $y_1, \cdots, y_p$) : if $\omega = \sum_{j=1}^p \omega_j dy_j$ is 1-form on $\mathcal{O}$ (hence the ω_j are functions on $\mathcal{O}$), its pull back by F is the 1-form on Ω

$$F^*\omega = \sum_{j=1}^p (\omega_j \circ F) dF_j = \sum_{i=1}^n \sum_{j=1}^p (\omega_j \circ F) \frac{\partial F_j}{\partial x_i} dx_i.$$

7.3 Integration of 1-forms

If $\varpi = u(t)dt$ is a 1-form on an interval $[a, b] \subset \mathbb{R}$, one defines its integral over the oriented interval $[a, b]$ by

$$\int_{[a,b]} \varpi = \int_a^b u(t)dt.$$

Now, if $\omega = \sum_{i=1}^n \omega_i dx_i$ and $c : [a, b] \to \Omega$ are respectively a 1-form and an oriented differentiable path on Ω, one defines *the integral of* ω *on the oriented path* c by

$$\int_c \omega := \int_a^b c^*\omega \, ,$$

which is nothing but the classical line integral $\int_c \sum_{i=1}^n \omega_i(x_1, \cdots, x_n)dx_i$.
The reader will show that if $\varphi : [\alpha, \beta] \to [a, b]$ is an increasing diffeomorphism, $\int_\alpha^\beta \varphi^*\varpi = \int_a^b \varpi$ and hence that the definition just given of $\int_c \omega$ is independent of the choice of the parametrization of the path c provided this parametrization respects orientation (i.e. it does not change if c is replaced by $c \circ \varphi$ where φ respects orientation). Hence integration is a duality pairing between 1-forms and oriented paths.

In the sequel, we define k-forms for $k \geq 2$; we need first recall some algebra:

7.4 Exterior forms

A k-linear antisymmetric form on a real vector space E is a mapping $f : E^k \to \mathbb{R}$ linear in each of its k arguments and such that, for every permutation σ of the set $\{1, 2, \cdots, k\}$, one has

$$f(x_1, x_2, \cdots, x_k) = \epsilon(\sigma) f(x_1^\sigma, x_2^\sigma, \cdots, x_k^\sigma),$$

where $\epsilon(\sigma) = \pm 1$ is the signature of σ and we noted $x_i^\sigma = x_{\sigma(i)}$. The set of these forms has a natural structure of a real vector space ; it is noted $\wedge^k E^*$. If

n is the dimension of E, n, the dimensions of $\wedge^k E^*$ is $\binom{n}{k}$. In particular, every exterior n-form on $\mathbb{R}^n$ is a multiple of the determinant and orienting E amounts to choosing which half-line of $\wedge^n E \cong \mathbb{R}$ will be called positive.

Definition 37 *Let f be an element of $\wedge^k E^*$, g an element of $\wedge^l E^*$, and v an element of E. The* interior product *$i_v f$ of f by v is the element of $\wedge^{k-1}E^*$ defined by*

$$(i_v f)(x_1, x_2, \cdots, x_{k-1}) = f(v, x_1, x_2, \cdots, x_{k-1}).$$

The exterior product *$f \wedge g$ of f by g is the element of $\wedge^{k+l}E^*$ defined by*

$$(f \wedge g)(x_1, \cdots, x_{k+l}) = \sum_{\sigma \in \Sigma} \epsilon(\sigma) f(x_1^\sigma, \cdots, x_k^\sigma) g(x_{k+1}^\sigma, \cdots, x_{k+l}^\sigma),$$

*where Σ is the set of (k,l)-*threshing*, i.e. the set of permutations σ of the set $\{1, 2, \cdots, k+l\}$ such that $\sigma(1) < \cdots < \sigma(k)$ and $\sigma(k+1) < \cdots < \sigma(k+l)$.*

The exterior product is associative, $(f \wedge g) \wedge h = f \wedge (g \wedge h)$, distributive, $\forall \lambda, \mu \in R, \quad (\lambda f + \mu g) \wedge h = \lambda(f \wedge g) + \mu(f \wedge h)$, and anticommutative, $f \wedge g = (-1)^{kl} g \wedge f$. The above definition of the exterior product as the "anti-symmetrized" of the tensor product becomes simpler when f and g are 1-forms: let $f_1, \cdots, f_k$ be elements of $\wedge^1 E^* = E^*$; the exterior product becomes a determinant (associativity allows to suppress parentheses) :

$$(f_1 \wedge \cdots \wedge f_k)(x_1, \cdots, x_k) = \det \begin{bmatrix} f_1(x_1) & \cdots & f_1(x_k) \\ \cdots & \cdots & \cdots \\ f_k(x_1) & \cdots & f_k(x_k) \end{bmatrix}.$$

Let $\{e_1, \cdots, e_n\}$ be a basis of E, $\{e_1^*, \cdots, e_n^*\}$ the dual basis. Computing dimensions show immediately the following

Lemma 38 *If the dimension of E is n, the collection of elements $e_{i_1}^* \wedge \cdots \wedge e_{i_k}^*$ such that $1 \le i_1 < \cdots < i_k \le n$, is a basis of $\wedge^k E^*$.*

The above lemma allows easy computation of the exterior product of two exterior forms f and g of respective degrees k and l: once chosen a basis of E, one decomposes the two forms on the bases of $\wedge^k E^*$ and $\wedge^l E^*$ given by the lemma and one applies associativity, distributivity, and anticommutativity to reorder the terms. One obtains the expression of $f \wedge g$ on the basis of $\wedge^{k+l}E^*$ given by the lemma.

7.5 Differential k-forms

In the same way as a 1-form ω on the open set Ω of $\mathbb{R}^n$ consists in giving for each q in Ω an element $\omega(q)$ of the cotangent space $T_q^*\Omega$ depending differentiably on q, a k-form ω on Ω consists in giving for each q an element $\omega(q)$, depending differentiably on q, of $\wedge^k (T_q\Omega)^* = \wedge^k T_q^*\Omega$. The meaning of such a "differentiable dependence" is clear as soon as one canonically identifies each tangent space

$T_q\mathbb{R}^n$ to $\mathbb{R}^n$ by translation. A differential k-form on Ω of class C^r is then identified to a C^r mapping from ω to $\wedge^k(\mathbb{R}^n)^*$.

Coupling with vector fields. The result $\omega(X_1,\cdots,X_k)$ of the coupling of a differential k-form with a k-tuple of vector fields is a function on Ω : if each vector field is identified with a mapping X_i from Ω to $\mathbb{R}^n$, one has $\omega(X_1,\cdots,X_k)(q) = \omega(q)\big(X_1(q),\cdots,X_k(q)\big)$. Note that this coupling is $C^\infty(\Omega,R)$-linear in each of its arguments: this is the manifestation of the tensorial character of differential forms (see appendix 2), that is of the fact that the value of $\omega(X_1,\cdots,X_k)$ at some point depends only of the values of $X_1,\cdots,X_k$ at this point. Conversely, one shows that a mapping ω, which to an ordered k-tuple $X_1,\cdots,X_k$ of vector fields on Ω associates a function $\omega(X_1,\cdots,X_n)$ of classe C^∞ onr Ω, is defined by a k-differential form if and only if it is $C^\infty(\Omega,R)$-linear in each of its arguments and antisymmetric.

Interior and exterior products.

Being defined at each point, the interior and exterior product keep their meaning for vector field and differential forms: the interior product $i_X\omega$ is a $(k-1)$-form if ω is a k-form and X a vector field; the extrior product $\omega_1\wedge\omega_2$ is a $(k+l)$-form if ω_1 is a k-form and ω_2 is a l-form.

Canonical expression.

The derivative $d\varphi$ of a function $\varphi:\Omega\to\mathbb{R}$, that is the family of the derivatives $d\varphi(q)$ of φ at each point $q\in\Omega$, is a differential 1-form on Ω. Recall that the 1-form dq_i is the derivative of the "i-th" coordinate" function $(q_1,\cdots,q_n)\mapsto q_i$; one deduces from lemma 38 that each k-form of class C^r on Ω can be uniquely written

$$\omega = \sum_{i_1<\cdots<i_k} a_{i_1\cdots i_k}\, dq_{i_1}\wedge\cdots\wedge dq_{i_k},$$

where the $a_{i_1\cdots i_k}$ are functions of class C^r on Ω.

Pullback and integration.

Definition 39 *Let $\mathcal{O}\subset\mathbb{R}^m$ and $\Omega\subset\mathbb{R}^n$ be two open subsets, $\varphi:\mathcal{O}\to\Omega$ a differentiable mapping, ω a differential k-form on Ω. The pullback $\varpi=\varphi^*\omega$ of ω by φ is the differential k-form on $\mathcal{O}$ defined, for any $x\in\mathcal{O}$ and $\xi_1,\cdots,\xi_k\in T_x\mathcal{O}$, by*

$$\varpi(x)(\xi_1,\cdots,\xi_k) = \omega(\varphi(x))\big(d\varphi(x)\xi_1,\cdots,d\varphi(x)\xi_k\big),$$

As in the case of 1-forms, if c is differentiable map from the k-dimensional cube $[0,1]^k$ into Ω and ω is a k-form on ω, one defines

$$\int_c \omega := \int_{[0,1]^k} c^*\omega := \int_{[0,1]^k} \sum_{i_1<\cdots<i_k} a_{i_1\cdots i_k}\, dq_{i_1}\cdots dq_{i_k},$$

where the k-form on the cube $c^*\omega$ is $c^*\omega = \sum_{i_1<\cdots<i_k} a_{i_1\cdots i_k} dq_{i_1}\wedge\cdots\wedge dq_{i_k}$. One checks that it is well defined up to composition of c with a diffeomorphism of the cube which preserves orientation (i.e. a diffeomorphism whose derivative at each point has a positive determinant; indeed, when dealing with n-forms instead of measures, $|\det\varphi|$ is replaced by $\det\varphi$ in the formulas).

7.6 Boundary and coboundary

Integration of differential forms provides a natural setting for generalizing the fondamental theorem of the differential and et integral calculus, that is the (so-called) Stokes formula

$$\int_c d\omega = \int_{\partial c} \omega,$$

which, rather than a theorem, is a definition of the coboundary $\omega \mapsto d\omega$ of a k-form as the adjoint of the boundary $c \mapsto \partial c$ of an oriented singular chain (that is of a formal sum of differentiable images of oriented submanifolds with boundary or corners) of dimension k :

$$< c, d\omega >=< \partial c, \omega > .$$

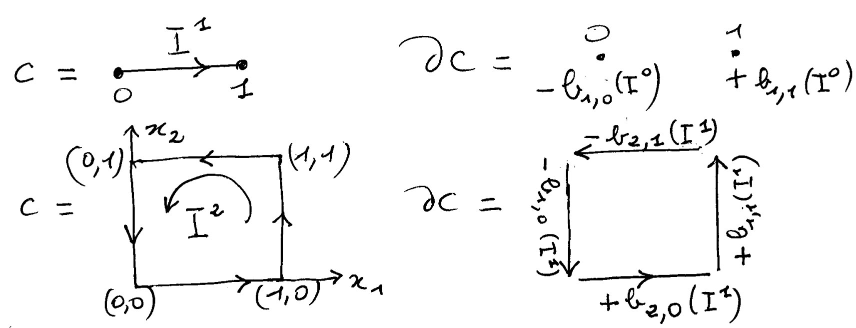
A natural way of making use of this definition to find an expression of the coboundary is to consider the integral of a differential $(n-1)$-form on the oriented boundary of an n-cube:

The oriented boundary – a formal sum of oriented faces – of an oriented cube $I^n := [0,1]^n$ (and in the same way, the definition of the oriented boundary ∂c of an oriented singular chain c), is defined by the following formula (a hat over a letter means that the letter is absent):

$$\partial I^n = \sum_{i=1}^{n} (-1)^i \left[b_{i,0}(I^{n-1}) - b_{i,1}(I^{n-1}) \right], \quad \text{where}$$

$$\begin{cases} b_{i,0}(x_1, \cdots, \hat{x}_i, \cdots, x_n) = (x_1, \cdots, x_{i-1}, 0, \cdots, x_n), \\ b_{i,1}(x_1, \cdots, \hat{x}_i, \cdots, x_n) = (x_1, \cdots, x_{i-1}, 1, \cdots, x_n). \end{cases}$$

The formula becomes clear when looking at the low dimensional cases:



If ω is an $(n-1)$-form on $\mathbb{R}^n$, we define its integral on the boundary of I^n by linearity, i.e.

$$\int_{\partial I^n} \omega := \sum_{i=1}^{n} (-1)^i \int_{I^{n-1}} \left(b_{i,0}^* \omega - b_{i,1}^* \omega \right).$$

For computing, it will be convenient to use the notation

$$\omega = \sum_{i=1}^{n} \omega_i \quad \text{with} \quad \omega_i = f_i(x_1, \cdots, x_n) dx_1 \wedge \cdots \wedge \hat{dx_i} \wedge \cdots \wedge dx_n,$$

and notice that for $\epsilon = 0$ or 1,

$$b_{i,\epsilon}^* \omega_j(x_1, \cdots, \hat{x_i}, \cdots, x_n) = \begin{cases} 0 \quad \text{if} \quad i \neq j, \\ f_i(x_1, \cdots, \epsilon, \cdots, x_n) dx_1 \wedge \cdots \wedge \hat{dx_i} \wedge \cdots \wedge dx_n \quad \text{if } i = j, \end{cases}$$

where the ϵ is at place i.
Hence, grouping opposite faces and applying the fundamental theorem of differential and integral calculus, we get the following expressions for $\int_{\partial I^n} \omega$:

$$\begin{aligned}
&\sum_{i=1}^{n} (-1)^i \int_{I^{n-1}} \left[f_i(x_1, \cdots, 0, \cdots, x_n) - f_i(x_1, \cdots, 1, \cdots, x_n) \right] dx_1 \wedge \cdots \wedge \hat{dx_i} \wedge \cdots dx_n \\
&= \sum_{i=1}^{n} (-1)^{i-1} \left(\int_0^1 \frac{\partial f_i}{\partial x_i}(x_1, \cdots, x_n) dx_i \right) dx_1 \wedge \cdots \wedge \hat{dx_i} \wedge \cdots \wedge dx_n \\
&= \sum_{i=1}^{n} (-1)^{i-1} \left(\int_0^1 \frac{\partial f_i}{\partial x_i}(x_1, \cdots, x_n) dx_i \right) dx_1 \cdots \hat{dx_i} \cdots dx_n \\
&= \sum_{i=1}^{n} (-1)^{i-1} \int_{I^n} \frac{\partial f_i}{\partial x_i}(x_1, \cdots, x_n) dx_1 \cdots dx_n \\
&= \sum_{i=1}^{n} (-1)^{i-1} \int_{I^n} \frac{\partial f_i}{\partial x_i}(x_1, \cdots, x_n) dx_1 \wedge \cdots \wedge dx_n \\
&= \sum_{i=1}^{n} \int_{I^n} \frac{\partial f_i}{\partial x_i}(x_1, \cdots, x_n) dx_i \wedge dx_1 \wedge \cdots \wedge \hat{dx_i} \wedge \cdots \wedge dx_n \\
&= \int_{I^n} \sum_{i=1}^{n} df_i(x_1, \cdots, x_n) \wedge dx_1 \wedge \cdots \wedge \hat{dx_i} \wedge \cdots \wedge dx_n.
\end{aligned}$$

In the last two lines we have used antisymetry first in the form $dx_i \wedge dx_j = -dx_j \wedge dx_i$, then in the form $dx_j \wedge dx_j = 0$. Finally, we are led to define

$$\begin{aligned}
&d\left(\sum_{i=1}^{n} f_i(x_1, \cdots, x_n) dx_1 \wedge \cdots \wedge \hat{dx_i} \wedge \cdots \wedge dx_n \right) \\
&= \sum_{i=1}^{n} df_i(x_1, \cdots, x_n) \wedge dx_1 \wedge \cdots \wedge \hat{dx_i} \wedge \cdots \wedge dx_n.
\end{aligned}$$

or, abandonning the convenient notation we had chosen for the form ω,

$$d\left(\sum \omega_{i_1 \cdots i_k} dx_{i_1} \wedge \cdots \wedge dx_{i_k} \right) = \sum \left(d\omega_{i_1 \cdots i_k} \right) \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k} .$$

Of course, this is a purely computational formula in coordinates which does not make clear what is the action of $d\omega$ on an arbitrary $(k+1)$-tuple of vector fields and hence hides the fundamental relation between coboundary and bracket. Nevertheless, before looking at the next proposition, the reader may check that coboundary is a natural operation i.e. that d commutes to pullbacks :

$$\varphi^* d\omega = d(\varphi^*\omega).$$

Proposition 40 *The coboundary of a differential k-form ω satisfies:*

$$\begin{array}{l} d\omega(X_1,\cdots X_{k+1}) = \sum_{i=1}^{k+1}(-1)^{i+1}\partial_{X_i}\big(\omega(X_1,\cdots,\hat{X}_i,\cdots,X_{k+1})\big) \\ +\sum_{1\le i<j\le k+1}(-1)^{i+j}\omega\big([X_i,X_j],X_1,\cdots,\hat{X}_i,\cdots,\hat{X}_j,\cdots,X_{k+1}\big), \end{array}$$

where $X_1,\cdots X_{k+1}$ are vector fields and as usual the hat indicates the absence of the term which wears the hat. In particular, if α is a differential 1-form and X, Y are vector fields,

$$d\alpha(X,Y) = \partial_X\big(\alpha(Y)\big) - \partial_Y\big(\alpha(X)\big) - \alpha([X,Y]).$$

Proof. One shows that the second term of the formula is $C^\infty(\Omega, R)$-linear in each of its arguments and that it coincides with the lexpression of $d\varpi$ in coordinates, that is when the X_i are constant vector fields of the form $\dfrac{\partial}{\partial x_j}$.

7.7 Lie derivative and the Cartan formula

The Lie derivative of the differential form ω along the vector field X may be defined by the formula

$$L_X\omega = \frac{d}{dt}\big(\varphi_t^*\omega\big)\Big|_{t=0},$$

in which φ_t stands for the local flow of X. It vanishes identically if and only if ω is invariant under this flow, i.e. if for all t, one has $\varphi_t^*\omega = \omega$. This definition is the same as the one for functions ($L_X f = \partial_X f$) and vector fields ($L_X Y = [X,Y]$).

Interior product, Lie derivative and coboundary are linked by Cartan's homotopy formula (see ....),

$$L_X\omega = i_X d\omega + d i_X\omega.$$

This is the infinitesimal version for the map $(x,t)\to\varphi_t(x)$ (φ_t is the flow of X) of the homotopy formula which describes the structure of differential forms on the product of a manifold by an interval. From this formula follows the commutation of Lie derivative with coboundary. Note that the formula expressing the value of the coboundary of a differential 1-forme α on a couple of vector fields X, Y can be seen as a Leibniz formula for the Lie derivative:

$$L_X\big(\alpha(Y)\big) = (L_X\alpha)(Y) + \alpha(L_X Y),$$

Finally, recall that a volume form π on a manifold M is invariant under the flow of a vector field X if and only if the divergence $\mathrm{div}_\pi X$ of X with respect to π, defined by $d(i_X\pi) = (div_\pi X)\pi$, identically vanishes.

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